Yerevan Physics Institute

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Dualities and Defects

PhD Thesis

01.04.02 - Theoretical Physics

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Yerevan-2017

Contents

0.1 Introduction

Two dimensional quantum field theories which are invariant under conformal transformations are referred as two dimensional conformal field theories (CFT). The applications of two dimensional CFTs to various topic of physics are numerous, here we will list two of the most important ones.

The first branch of application of CFTs is in statistical physics. Historically conformal symmetry was introduced in quantum field theory nearly fifty years ago under the influence of ideas of scaling and universality in the theory of second-order phase transitions. According to the scaling postulate at the critical point the interaction of fields corresponding to the order parameters of transitionally invariant and isotropic statistical systems become scale invariant. The energy-momentum tensor of such theories is traceless. As a consequence this kind of theories are also invariant with respect to a larger class of coordinate transformations under which metric tensor gets multiplied by an arbitrary function. Such coordinate transformations form the conformal group.

The second branch of applications of CFTs is string theory. It is well known that string theory is the most well developed candidate which may unify all known interactions included gravity. In this context CFT describes the world sheet dynamics of a string.

This dissertation is organized in the following way.

The dissertation consists of 6 chapters. In chapter 1 we review the material necessary to present our findings. In chapters 2-6 we deliver our findings.

In chapter 1 we collect and review the basik stuff of two-dimensional CFT. In section 1.1 we review two-dimensional conformal field theory, in particular we show that the generators of conformal transformations obey the Witt algebra. In section 1.2 we study: The energy-momentum tensor, radial quantization, OPE of operators and two, three- point functions. In section 1.3 we examine the Virasoro algebra and illustrate the construction of the Verma module.

The Chapter 2 is based on the paper [4].

The existence of a RG flow between two CFT's suggests that this theories could be connected by a non-trivial interface which encodes the map from the UV observables to the IR ones [8, 9] In particular in [9] such an interface (RG domain wall) was constructed for the $N = 2$ superconformal models using matrix factorisation technique.

Later in [10] an algebraic construction of a RG domain wall for the unitary minimal CFT models was proposed and was shown that the results agree with those of the leading order perturbative analysis performed by A. Zamolodchikov in [11]. The leading order perturbative calculation of the mixing coefficients for the wider class of local fields including non-primary ones again is in an impressive agreement with the RG domain wall approach [12]. Higher order perturbative calculations [13, 14] further confirm the validity of this construction. In the same paper [10] Gaiotto suggests that a similar construction should be valid also for more general coset CFT models. The $N = 1$ minimal superconformal CFT models [130–132], which are the main subject of this paper, are among these cosets. The Renormalisation Group (RG) flow between minimal $N = 1$ superconformal models SM_p and SM_{p-2} initialised by the perturbation with the top component of the Neveu-Schwarz superfield $\Phi_{1,3}$ in leading order of the perturbation theory has been investigated in [18] (see also [19, 20]). Recently, extending the technique developed in [13] for the minimal models to the supersymmetric case, in [21] the analysis of this RG flow has been sharpened even further by including also the next to leading order corrections.

In this chapter we specialise Gaiotto's proposal to the case of the minimal N=1 SCFT models. The method we use is based directly on the current algebra construction and, in this sense, is more general than the one originally employed by Gaiotto for the case of minimal models. Namely he heavily exploited the fact that the product of successive minimal models can be alternatively represented as a product of $N = 1$ superconformal and Ising models. We explicitly calculate the mixing coefficients for several classes of fields and compare the results with the perturbative analysis of [18, 21] finding a complete agreement.

It is organized as follows: section 2.2.2 is a brief review of the 2d $N = 1$ super-conformal filed theories. Section 2.2.3 is devoted to the description of the coset construction of $N = 1$ SCFT. In section 2.2.4 we formulate Gaiotto's general proposal for a class of coset CFT models. Section 2.2.5 is the main part of our paper. We explicitly calculate the mixing coefficients for the several classes of local fields in the case of the super-symmetric RG flow discussed above using RG domain wall proposal. Then we compare this with the perturbation theory results available in the literature finding a complete agreement.

The chapter 3 is based on the paper [100]. It is organized in the following way.

In section 3.1 we analyze classical Liouville theory with defects. In subsection 3.1.1 we review general solutions of the Liouville equation. In subsection 3.1.2 we present general solution of the defect equations of motion. In subsection 3.1.3 we present Lagrangian of the product of the Liouville theories on half-plane with the boundary condition specified by a permutation brane. In section 3.2 we review defects and permutation branes in quantum Liouville theory. In section 3.3 we review heavy and light asymptotic semiclassical limits. In section 3.4 we calculate defect two-point function in the light asymptotic limit. In section 3.5 we calculate defect two-point function in the heavy asymptotic limit.

The chapter 4 is based on the paper [5].

During the last decades we got deep understanding of the properties of rational CFTs having a finite number of primaries. Many important relations were obtained between basic notions of RCFT. In particular we would like to mention the Verlinde formula [50], relating matrix of modular transformation and fusion coefficients, Moore-Seiberg relations between elements of fusion matrix, braiding matrix and matrix of modular transformations [51–53]. We have formulas for boundary states [54], and defects [55, 56] in rational conformal field theories.
Situation in non-rational CFTs is much more complicated. The infinite and even uncountable number of primary fields is the main reason that progress in this direction is very slow. One of the well studied non-rational theories is Liouville field theory. Here three-point correlation function (DOZZ formula) [57, 95] and fusing matrix [59, 139] were found exactly. An other important examples of the non-rational CFT is $N = 1$ superconformal Liouville theory. Many data have been collected also in $N = 1$ superconformal Liouville theory. In particular three-

point functions [69, 70] and the NS sector fusion matrices [71, 72] have been found exactly.

In this paper we study some of the Moore-Seiberg relations for the fusion matrix of the N=1 Super Liouville field theory. Recall some basic facts on the fusion matrix. It is defined as a matrix of transformation of conformal blocks [119] in s and t channels [53]:

$$\mathcal{F}^s_p \begin{bmatrix} k & j \\ & \\ i & l \end{bmatrix} = \sum_q F_{p,q} \begin{bmatrix} k & j \\ & \\ i & l \end{bmatrix} \mathcal{F}^t_q \begin{bmatrix} l & j \\ & \\ i & k \end{bmatrix} . \tag{1}$$

Here we write all formulas in the absence of the multiplicities *i.e.* for the fusion numbers $N^i_{jk} = 0, 1$. Fusion matrix plays an important role in conformal field theories, *e.g.* it enters in the conformal bootstrap [53, 74], and Cardy-Lewellen [**?**] equations.

Our task here is to study the following relations, proved in rational CFT, in $N = 1$ super Liouville field theory:

$$F_{0,i} \begin{bmatrix} j & k \\ & \\ j & k^* \end{bmatrix} F_{i,0} \begin{bmatrix} k^* & k \\ & \\ j & j \end{bmatrix} = \frac{F_j F_k}{F_i} , \tag{2}$$

where

$$F_i \equiv F_{0,0} \begin{bmatrix} i & i^* \\ & \\ i & i \end{bmatrix} = \frac{S_{00}}{S_{0i}} . \tag{3}$$

and

$$C^p_{ij} = \frac{\eta_i \eta_j}{\eta_0 \eta_p} F_{0,p} \begin{bmatrix} j & i \\ & \\ j & i^* \end{bmatrix} , \qquad \eta_i = \sqrt{C_{ii^*}/F_i} , \tag{4}$$

which using (2) can be written also as

$$C^p_{ij} = \frac{\xi_i \xi_j}{\xi_0 \xi_p} \frac{1}{F_{p,0} \begin{bmatrix} j^* & j \\ & \\ i & i \end{bmatrix}} , \qquad \xi_i = \eta_i F_i = \sqrt{C_{ii^*} F_i} . \tag{5}$$

Let us explain notations. First of all 0 denotes vacuum field and i^* is the field conjugate to i in a sense that $N^0_{ii*} = 1$. Then S_{ij} is a matrix of the modular transformations, C^p_{ij} are structure constants, C_{ii^*} are two-point functions.

The relation (2) is a consequence of the pentagon identity for fusion matrix [51–53]. The expression (3) results from the two different ways of calculation of the quantum dimension [52]. The equations (4) and (5) result from the bootstrap equation combined with the pentagon identity [54, 74–76].

These relations were examined in the Liouville field theory. The eq.(2) in the Liouville field theory was tested in [77]. The expressions (4) and (5) were examined in the Liouville field theory in [76, 78]. In [76], (4) and (5) in the Liouville field theory were checked using the relation of the fusion matrix with boundary three-point function. In [78], eq.(4) was checked using the following star-triangle integral identity for the double Sine-functions $S_b(x)$:

$$\int \frac{dx}{i} \prod_{i=1}^{3} S_b(x+a_i) S_b(-x+b_i) = \prod_{i,j=1} S_b(a_i+b_j)\,, \tag{6}$$

where

$$\sum_i (a_i+b_i) = Q\,. \tag{7}$$

Recently it was found in [81] the supersymmetric generalization of this formula (eq.(4.56) in text).

Our first aim here is to calculate the elements of the fusion matrix in the NS sector constructed in [71, 72] with one of the intermediate entries set to the vacuum. For this purpose we find convenient to define general expressions for the fusion matrix and structure constants, composed from the supersymmetric double Gamma and double Sine-functions, which reduce to the known elements of the NS sector fusion matrix and structure constants for the certain choices of the types of the supersymmetric double functions. Using the supersymmetric version of the star-triangle identity (4.56) we found constraints which should be satisfied by the types of the supesymmetric double functions to ensure that the elements of the fusion matrix with one of the entries set to the vacuum give rise to the corresponding structure constant according to the pattern of the equations (4) and (5). We checked that the elements of the fusion matrix in the NS sector indeed satisfy these constraints, and thus established equations (4) and (5) for the NS sector of the N=1 Super Liouville field theory.

Next we turn to the fusion matrix in the Ramond sector. Since the general expression for fusion matrix in the Ramond sector is absent, we check the equations (4) and (5) for the elements of the fusion matrix with a degenerate entry, computed in [79, 80]. Setting the intermediate state to the vacuum we find that at least these particular elements of the fusion matrix in the Ramond sector again satisfy (5). This drastically simplifies the Cardy-Lewllen equations. It enables us easily to construct topological defects in the N=1 super Liouville field theory. In section 4.1 we review basic facts on $N = 1$ super-Liouville theory. In section 4.2 we compute the elements of an Ansatz for the fusion matrices with one of intermediate states set to the vacuum state. In section 4.3 we specialize the formulae obtained in section 4.2 to the fusion matrices of the NS sector found in [71]. In section 4.4 we analyze the Ramond sector. In section 4.5 we apply formulae obtained in section 4.4 to solve the Cardy-Lewellen equations for topological defects.

The chapter 5 is based on the paper [6].

Semiclassical limits play important role since they link quantum physics to the Lagrangian approach. In the Liouville and Toda field theories there are three semiclassical limits: minisuperspace [90–93], the light and heavy [93–95]. All three asymptotics are the large central charge limits. The difference comes in the treatment of the primary fields. In the minisuperspace limit one considers a limit where only the zero mode dynamics survives. In this limit the Liouville and Toda field theories reduce to the corresponding quantum mechanical problems [90, 91, 93]. In the light asymptotic limit one keeps the conformal dimensions fixed. Then the correlation functions are given by the finite dimensional path integral over solutions of the equations of motion with a vanishing energy-momentum tensor. And finally in the heavy asymptotic limit the conformal dimensions blow up, scaling as the classical action and correlation functions are given by the exponential of the action evaluated over the singular solutions. To be more specific recall that primary fields in the Liouville and Toda field theories are related to the vertex operators $V_\alpha = e^{i\alpha\phi}$. The spectrum is given by $\alpha = \frac{Q}{2} + iP$. In the light asymptotic limit we set $\alpha = \eta_l b$ and keep η_l fixed for $b \to 0$, whereas in the heavy asymptotic limit we take $\alpha = \frac{\eta_h}{b}$ and hold η_h fixed again for $b \to 0$. In the minisuperpsace limit one should take for some of the vertex operators $\alpha = \eta_m b$ and for some $P = \eta_m b$.

The discovery of AGT correspondence [102–104, 123] relating 2d CFT conformal blocks to the Nekrasov partition function [105, 124] in $\mathcal{N} = 2$ supersymmetric gauge theory provides powerful tools to investigate CFT correlators using gauge theory methods or alternatively to apply advanced CFT methods in gauge theory (see e.g. [107,108]). The essential point here is the fact that there are explicit combinatorial formulas for the Nekrasov partition function [110,127], which now can be successfully applied in 2d CFT.

In this chapter we consider the light asymptotic limit of the $U(n)$ Nekrasov partition functions for an arbitrary n. We find that for the certain choice of fields the Nekrasov partition functions in the light asymptotic limit are simplified drastically and given by the sum over Young diagrams having at most $n - 1$ rows. We compute the corresponding W_3 conformal block using the light asymptotic integral representation and found perfect agreement with the two-row Nekrasov partition functions. Note that in the light asymptotic limit the W_n symmetry reduces to $SL(n)$ group [115, 118] and this already hints on the existence of the limiting procedure where survive only Young diagrams corresponding to the $SL(n)$ representations.

In section 5.1 we compute the light asymptotic limit of the Nekrasov partition functions. In subsection 5.1.1 we review the necessary facts on the Nekrasov partition functions. In subsection 5.1.2 we review Toda conformal field theory and the AGT relation. In subsection 6.3.2 we explain the details on the light asymptotic limit and show that choosing the data as it is specified in eq. (5.17) and (5.18) truncates the Nekrasov functions in the light asymptotic limit to the sum over Young tableaux containing at most $n-1$ rows. In subsection 5.1.4 we compute the Nekrasov partition function in the light asymptotic limit. The formula (5.33) is our main result. In section 5.2 we compute the corresponding conformal block in A_2 Toda field theory using that in the light asymptotic limit conformal blocks admit an integral representations.

The chapter 6 is based on the paper [7].

$\mathcal{N} = 1$ SLFT besides the spin two conserved currents (energy-momentum tensor) includes also spin 3/2 currents (the super-currents). These currents generate super conformal symmetry which in $2d$ is described by the Neveu-Schwarz-Ramond algebra [129,131,132]. If upon encircling a field by the super-current an extra multiplier -1 is produced, one refers to this field as a Ramond field. Those fields which are local with respect to the super current are called Neveu-Schwarz fields.

In this chapter different $\mathcal{N} = 1$ SLFT blocks in the light limit are derived by using the above mentioned duality between super Yang-Mills theory and $2d$ SCFT. We obtained that in the case of SLFT the analysis of the light limit is more subtle and complicated compare to the bosonic Lioville theory. In particular we found that in the light limit to the conformal blocks contribute not only one row diagrams. For instance the instanton partition functions that correspond to the conformal blocks with four Ramond fields also get contribution from the diagrams, like those in figures (6.3(b)) and (6.3(c)) below.

The paper is organized as follows. In section 6.1 the expression for the instanton partition functions of $\mathcal{N} = 2$ SYM on R^4/Z_2 [136, 137] is reviewed. In section 6.2 we bring known facts for $\mathcal{N} = 1$ SLFT and its light asymptotic limit that will be useful for us. In subsection 6.3.1 the map between $\mathcal{N} = 1$ super Liouville conformal blocks and $\mathcal{N} = 2$ SYM on R^4/Z_2 is given. In subsection 6.3.2 the rules for the light asymptotic limit are written. In section 6.4 we present new results on various partition function in the light limit. In section 6.5 by using these partition functions we give the corresponding conformal blocks in the light limit.

Chapter 1

Basics of conformal field theory in two dimensions

1.1 The two dimensional conformal group

Consider a diffeomorphism $f : x \mapsto x'$, where $x, x' \in M$ and M is a differentiable manifold. Suppose M is endowed with a metric $g_{\mu\nu}(x)$. Then one can construct another symmetric second rank tensor $g'_{\mu\nu}(x')$ such that $f_* g' = g$, i.e.

$$g_{\mu\nu}(x) = g'_{\lambda\rho}(x') \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} \, . \tag{1.1}$$

The map f is called conformal if the metric tensor satisfies

$$g_{\mu\nu}(x') = \Lambda(x') g'_{\mu\nu}(x'). \tag{1.2}$$

Since we are interested in the two dimensional flat metric, it follows from (1.1) and (1.2) that

$$g_{\lambda\rho} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} = \Lambda g_{\mu\nu} \, . \tag{1.3}$$

Now we want to examine the consequences of definition (1.2) on the infinitesimal level:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)\,. \tag{1.4}$$

The left hand side of (1.3) up to the first order in ϵ can be written as

$$g_{\lambda\rho}\left(\delta^\lambda_\mu + \partial_\mu\epsilon^\lambda\right)\left(\delta^\rho_\nu + \partial_\nu\epsilon^\rho\right) \approx g_{\mu\nu} + \partial_\nu\epsilon_\mu + \partial_\nu\epsilon_\mu\,. \tag{1.5}$$

Therefore the requirement that this map is conformal implies that

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = h(x)g_{\mu\nu}\,, \tag{1.6}$$

where $h(x)$ is some function that can be determined by taking trace on both sides of the last expression, which yields

$$h(x) = \frac{2}{d}\partial_\rho\epsilon^\rho\,. \tag{1.7}$$

For Euclidean metric i.e. $g_{\mu\nu} = diag(1,1)$, we can rewrite (1.6) as

$$\partial_1\epsilon_1 = \partial_2\epsilon_2\,; \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1\,, \tag{1.8}$$

which are the Cauchy-Riemann equations. A complex function whose real and imaginary parts satisfy the Cauchy-Riemann condition is a holomorphic function. Thus it is natural to introduce complex coordinates

$$\begin{aligned} z = x + iy\,; &\qquad \bar{z} = x - iy\,, \\ \epsilon(z) = \epsilon_1 + i\epsilon_2\,; &\qquad \bar{\epsilon}(\bar{z}) = \epsilon_1 - i\epsilon_2\,. \end{aligned} \tag{1.9}$$

Since $\epsilon(z)$ is holomorphic, the function $f(z) = z + \epsilon(z)$ is holomorphic too. So we can say that complex analytic coordinate transformations give rise to two dimensional conformal transformations. One could arrive to same conclusion by taking a different approach, namely by rewriting the metric in the complex coordinates: $ds^2 = dzd\bar{z}$. Indeed, under a holomorphic transformation $z \to f(z)$ this metric transforms as:

$$ds^2 = dzd\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z}\,. \tag{1.10}$$

Let us perform a Laurent expansion of $\epsilon(z)$. Then the infinitesimal conformal transformation can be written as

$$z' = z + \epsilon(z)\,; \qquad \epsilon(z) = \sum_{n\in\mathbb{Z}} c_n z^{n+1}\,, \tag{1.11}$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})\,; \qquad \bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{c}_n \bar{z}^{n+1}\,. \tag{1.12}$$

The operators that generate this transformations for a particular n are

$$l_n = -z^{n+1}\partial_z\,, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,. \tag{1.13}$$

These generators obey commutation relations:

$$[l_n, l_m] = (n-m)l_{n+m}\,, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}\,, \quad [l_n, \bar{l}_m] = 0\,, \tag{1.14}$$

the first and second commutation relations are two copies of the so called Witt algebra. As one can see from the last commutation relation the algebras $\{l_n\}$ and $\{\bar{l}_n\}$ can be regarded as independent from each other provided one treats z and $\bar{z}$ as independent variables. But this is just a complexification of the initial space: $\mathbb{C} \cong \mathbb{R}^2 \longmapsto \mathbb{C}^2$. Nevertheless at some point we have to identify $\bar{z}$ with z^*. From now on we will discuss the holomorphic dependence only and ignore the similar anti-holomorphic dependence.

In general, the generators l_n are not well defined everywhere and do not generate invertible transformations. Even on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \infty$, there are only few generators that are globally defined. Let us find them.

The analytic conformal transformations are generated by the vector fields:

$$v(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1} \partial_z \,. \tag{1.15}$$

The non-singularity of $v(z)$ as $z \to 0$ requires that $a_n \neq 0$ only if $n \geq -1$. To understand the behavior of $v(z)$ as $z \to \infty$, let us perform the transformation $z = -\frac{1}{\omega}$,

$$v(z) = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n-1} \partial_\omega \,. \tag{1.16}$$

The non-singularity as $\omega \to 0$ implies that $a_n \neq 0$ if $n \leq 1$. We conclude that only the subset $\{l_0, l_{\pm 1}\}$ generates conformal transformations that are globally defined on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \infty$. These generators satisfy the commutation relation:

$$[l_0, l_{-1}] = l_{-1}\,; \quad [l_0, l_1] = -l_1\,; \quad [l_1, l_{-1}] = 2l_0\,, \tag{1.17}$$

which is the $sl(2, \mathbb{C})$ algebra.

Let us examine also the group structure. Note that $l_0 = -z\partial_z$ and $\bar{l}_0 = -\bar{z}\partial_{\bar{z}}$ and hence introducing the polar coordinates $z = re^{i\theta}$ we obtain

$$r\frac{\partial}{\partial r} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = -(l_0 + \bar{l}_0)\,, \quad \frac{\partial}{\partial \theta} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} = -i(l_0 - \bar{l}_0)\,. \tag{1.18}$$

Thus $(l_0 + \bar{l}_0)$ generates dilatations and $i(l_0 - \bar{l}_0)$ generates rotations. From (1.13) it is obvious that:

- l_{-1} and $\bar{l}_{-1}$ are generators of translations (globally $z \to z + \alpha$);
- l_0 and $\bar{l}_0$ are generators of dilatations (globally $z \to \lambda z$);
- l_1 and $\bar{l}_1$ are generators of the special conformal transformations (globally $z \to \frac{z}{1-\beta z}$).

Together these transformations form a group known as the complex Möbius group:

$$z \to \frac{az+b}{cz+d}\,, \tag{1.19}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. This is the group $SL(2,\mathbb{C})/\mathbb{Z}_2$. The quotient by $\mathbb{Z}_2$ is due to the fact that (1.19) is unchanged uder simultaneous flip of signs of the parameters a, b, c, d.

1.2 The energy-momentum tensor, radial quantization, OPE of operators and two, three- point functions

Energy-momentum tensor

Here we want to find the constraints on the energy-momentum tensor that are due to the conformal symmetry $x^\mu \to x^\mu + \epsilon^\mu(x)$ of our theory. Under this coordinate transformation the action changes in the following way:

$$\delta S = \int d^2x T^{\mu\nu}\partial_\mu \epsilon_\nu = \frac{1}{2}\int d^2x T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{1.20}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor. The definition (1.6) of the infinitesimal conformal mapping implies that corresponding variation of the action reads

$$\delta S = \frac{1}{2}\int d^2x T^\mu_\mu \partial_\rho \epsilon^\rho \tag{1.21}$$

The vanishing of the trace of the energy-momentum tensor thus implies the invariance of the action under the conformal transformation. The conserved current of conformal symmetry can be written as

$$j_\mu = T_{\mu\nu}\epsilon^\nu(x). \tag{1.22}$$

Lets go back to CFTs with Euclidean signature. The metric in the complex coordinates has the form (1.10). Obviously $g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. We will show that the energy-momentum tensor has two non-vanishing components and that one of them is holomorphic while the other is antiholomorphic. By using $T_{\mu\nu} = \frac{\partial x^\lambda}{\partial x^\mu}\frac{\partial x^\rho}{\partial x^\nu}T_{\lambda\rho}$ we can express the components of the energy-momentum tensor in the complex coordinates in terms of they initial Euclidean components:

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})\,; \quad T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})\,; \tag{1.23}$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^\mu_\mu\,.$$

Therefore the tracelessness implies

$$T_{z\bar{z}} = T_{\bar{z}z} = 0\,. \tag{1.24}$$

The conservation law $\partial^\mu T_{\mu\nu} = 0$ gives:

$$\begin{array}{lcl} \partial_{\bar{z}}T_{zz} + \partial_z T_{\bar{z}z} = 0; & \Rightarrow & \partial_{\bar{z}}T_{zz} = 0, \\ \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}}T_{z\bar{z}} = 0; & & \partial_z T_{\bar{z}\bar{z}} = 0. \end{array}$$

We see that the two non-vanishing components of the energy-momentum tensor

$$T(z) \equiv T_{zz}(z) \qquad \text{and} \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \tag{1.25}$$

have holomorphic and anti-holomorphic dependence on their arguments respectively.

To avoid infrared divergences we compactify the space coordinate. Thus we consider our system to live on a cylinder $\Sigma = R \times S^1 = (\sigma^0, \sigma^1 \bmod 2\pi)$, where $\sigma^0 \in \mathbb{R}$ is the Euclidean time and σ^1 is the compactified space coordinate. Then we can go back to the complex plain by the exponential map

$$z = e^w\,, \quad w = \sigma^0 + i\sigma^1\,. \tag{1.26}$$

The infinite past and future on a cylinder, $\sigma^0 = -\infty\,, \infty$ are mapped to points $z = 0, \infty$ on a plane correspondingly. The equal time surfaces $\sigma^0 = \text{const}$ become circles of constant radii on

z-plane. Dilatation on the plane e^a becomes time translation $\sigma^0 + a$ on the cylinder, and rotation on the plane $e^{i\alpha}$ is space translation $\sigma^1 + \alpha$ on the cylinder. Therefore the dilatation generator on the conformal plane can be considered as the Hamiltonian, and the rotation generator as momentum.

As we see from (1.25) the current of conformal transformations is:

$$J_z = T(z)\epsilon(z) \quad \text{and} \quad J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\,. \tag{1.27}$$

The arguments we gave above make it reasonable to choose the radial arrays as time directions. Then the fixed time surfaces correspond to circles around the origin. So, the conserving charge of a conformal transformations is:

$$Q = \frac{1}{2\pi i}\oint dz T(z)\epsilon(z) + \frac{1}{2\pi i}\oint d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\,, \tag{1.28}$$

where the contour integrals are taken along circles mentioned above.

Radial ordering

In QFT correlation function are defined as a time ordered product. We know that passing from a cylinder to a plane, Euclidean time coordinate is mapped to radial coordinate, and the time ordering becomes the radial ordering. Thus it is reasonable to choose as the analog of time ordering on the complex plane radial ordering

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & \text{if } |z| > |w| \\ B(w)A(z) & \text{if } |z| < |w| \end{cases}.$$

The variation of any field generated by the conserved charge Q is given by the commutator with this charge. Making use of (1.28), we will get

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = [Q,\Phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz[\epsilon(z)T(z),\Phi(w,\bar{w})] + \frac{1}{2\pi i}\oint d\bar{z}[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\Phi(w,\bar{w})]. \tag{1.29}$$

We will discuss the holomorphic part, the antiholomorphic part is similar. In the expression above the products of operators is defined in the regions where the operators are radial ordered, thus:

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \tag{1.30}$$
$$\lim_{|z|\to|w|}\left(\frac{1}{2\pi i}\oint_{|z|>|w|} dz\epsilon(z)\left(T(z)\Phi(w,\bar{w})\right) - \frac{1}{2\pi i}\oint_{|z|<|w|} dz\epsilon(z)\left(\Phi(w,\bar{w})T(z)\right)\right).$$

We can rewrite this as

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \lim_{|z|\to|w|}\left(\frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\epsilon(z)R(T(z)\Phi(w,\bar{w}))\right). \tag{1.31}$$

One can deform the contours to get

$$\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \lim_{|z|\to|w|}\left(\frac{1}{2\pi i}\oint_w dz\epsilon(z)R(T(z)\Phi(w,\bar{w}))\right). \tag{1.32}$$

Obviously this integral does not vanish only if there is a singularity at the point w. Recollecting everything, we obtain that (1.29) is given

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \lim_{|z|\to|w|}\left(\frac{1}{2\pi i}\oint_w dz\epsilon(z)R(T(z)\Phi(w,\bar{w})) + \frac{1}{2\pi i}\oint_w d\bar{z}\bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(w,\bar{w}))\right). \tag{1.33}$$

Fields transforming under the conformal transformation $z \to f(z)$ according to

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \tilde{\Phi}(f(z),\bar{f}(\bar{z})), \tag{1.34}$$

are called primary fields with conformal dimension $(h\,,\bar{h})$. But if (1.34) is true for global conformal transformations only, then Φ is called a quasi-primary field.

Under infinitesimal conformal transformation $z \to z+\epsilon(z)$ primary fields of conformal weight $(h\,,\bar{h})$ transform as:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \left(\epsilon(w)\partial + h\partial\epsilon(w)\right)\Phi(w,\bar{w}) + \left(\bar{\epsilon}(\bar{w})\bar{\partial} + \bar{h}\bar{\partial}\bar{\epsilon}(\bar{w})\right)\Phi(w,\bar{w}). \tag{1.35}$$

Comparing (1.33) and (1.35) we get OPE of the energy-momentum tensor with the primary field of the weights $(h\,,\bar{h})$. afterwards we will omitted the R symbol and assume that products of operators are always radial ordered.

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) + \dots\,, \tag{1.36}$$
$$\bar{T}(\bar{z})\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) + \dots\,. \tag{1.37}$$

The operator product expansion between the energymomentum tensors and $\Phi(z,\bar{z})$ (1.36) and (1.37) is equivalent to (1.34) so it can be considered as the definition of a primary field $\Phi(z,\bar{z})$ with conformal dimensions $(h\,,\bar{h})$ as well.

Asymptotic States

Let us consider the field $\Phi(z,\bar{z})$, with conformal dimension $(h,\bar{h})$, its Laurent expansion around $z_0 = \bar{z}_0 = 0$ is

$$\Phi(z,\bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{m}-\bar{h}}\Phi_{n,\bar{m}}\,. \tag{1.38}$$

Since we have directed the time axis in the radial direction, the infinite past coincides with $z_0 = \bar{z}_0 = 0$ it is natural to define the in-states as:

$$|\Phi_{in}\rangle \equiv \lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})|0\rangle\,. \tag{1.39}$$

It follows from (1.38) that in order to get a well defined in-state the vacuum must satisfy the condition

$$\Phi_{n,\bar{m}}|0\rangle = 0 \quad \text{for all} \quad n > -h,\ \bar{m} > -\bar{h}\,. \tag{1.40}$$

On the Riemann sphere $\mathbb{S}^2 = \mathbb{C}\cup\infty$ the parametrization near ∞ is related to the one near the origin by the conformal map $z = 1/w$. Therefore, it is reasonable to introduce the out-state as follows:

$$\langle\Phi_{out}| = \lim_{w,\bar{w}\to 0}\langle 0|\tilde{\Phi}(w,\bar{w})\,, \tag{1.41}$$

where $\tilde{\Phi}(w,\bar{w})$ is the transformed field in w coordinates. For primary fields, applying (1.34) for the transformation $w \to z = 1/w$, one gets a relation between $\tilde{\Phi}(w,\bar{w})$ ant $\Phi(z,\bar{z})$:

$$\tilde{\Phi}(w,\bar{w}) = (w)^{-2h}(\bar{w})^{-2\bar{h}}\Phi\left(1/w, 1/\bar{w}\right)\,. \tag{1.42}$$

Inserting this into (1.41), we will get

$$\langle\Phi_{out}| = \lim_{z,\bar{z}\to\infty}\langle 0|\Phi(z,\bar{z})z^{2h}\bar{z}^{2\bar{h}}\,. \tag{1.43}$$

On the other hand

$$\langle\Phi_{out}| = |\Phi_{in}\rangle^{\dagger} = \left[\lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})|0\rangle\right]^{\dagger} = \lim_{z,\bar{z}\to 0}\langle 0|\left[\Phi(z,\bar{z})\right]^{\dagger}\,. \tag{1.44}$$

The consistency with (1.43) implies that

$$\left[\Phi(z,\bar{z})\right]^{\dagger} \equiv \Phi\left(\frac{1}{z},\frac{1}{\bar{z}}\right)\frac{1}{z^{2h}}\frac{1}{\bar{z}^{2\bar{h}}}\,. \tag{1.45}$$

Using the expansion (1.38) we get

$$\Phi^{\dagger}(z,\bar{z}) = z^{-2h}\bar{z}^{-2\bar{h}}\sum_{n,\bar{m}\in\mathbb{Z}} z^{n+h}\bar{z}^{\bar{m}+\bar{h}}\Phi_{n,\bar{m}} = \sum_{n,\bar{m}\in\mathbb{Z}} z^{n-h}\bar{z}^{\bar{m}-\bar{h}}\Phi_{n,\bar{m}}\,. \tag{1.46}$$

Comparing this result with (1.38), we will obtain:

$$\left(\Phi_{n,\bar{m}}\right)^{\dagger} = \Phi_{-n,-\bar{m}}\,, \tag{1.47}$$

By similar considerations, using (1.44) and (1.46) for the out-states we get

$$\langle 0|\Phi_{n,\bar{m}} = 0 \ \text{ for all } \ n < h, \ \bar{m} < \bar{h}\,. \tag{1.48}$$

Two and three point functions

The invariance under $SL(2,\mathbb{C})/\mathbb{Z}_2$ transformations determine the two and three- point functions of quasi-primary fields up to some constants.

For the two-point functions one gets

$$\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}\,\delta_{h_1,h_2}}{(z_{12})^{2h}(\bar{z}_{12})^{2\bar{h}}}\,, \tag{1.49}$$

where $z_{ij} \equiv z_i - z_j$, $h_1 = h_2 \equiv h$ and C_{12} are constants that can be absorbed into normalization of the fields. And for the three-point function the result is

$$\langle \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\Phi_3(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\,. \tag{1.50}$$

The numerical coefficients are important dynamical characteristics of the theory. Global conformal invariance does not fix the precise form of the four or higher point functions. We will discuss such correlation functions in detail later.

1.3 Virasoro algebra

Schwarzian derivative

Dimensional analysis and closedness condition predict the following general form for the OPE of the energy-momentum tensor with itself (cf. 1.36)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \dots\,, \tag{1.51}$$

where c is a numerical constant which is called the central charge or conformal anomaly. Its value, in general, will depend on the particular theory under consideration. The second term on the rhs of (1.51) indicates that $T(z)$ is a field with conformal weight $(2,0)$. According to (1.49) the two point correlation function of energy-momentum tensors is given by

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}\,. \tag{1.52}$$

Note that if we would not have a central extension term i.e. $c = 0$ then the two point correlation function (1.52) would vanish. Thus the energy-momentum tensor of our theory would always be zero. Thus the central extension term ensures the non triviality of our theory. According to (1.33) the variation of T under infinitesimal conformal transformation is

$$\delta_\epsilon T(w) = \frac{1}{2\pi i}\oint \epsilon(z)T(z)T(w) = \frac{1}{12}c\partial_w^3\epsilon(w) + 2T(w)\partial_w\epsilon(w) + \epsilon(w)\partial_w T(w)\,, \tag{1.53}$$

where we used the OPE of two energy-momentum tensors (1.51). One can exponentiate this and find how T transforms under a finite transformation $z \to w(z)$:

$$T(z) \to \left(\frac{dw}{dz}\right)^2 T(w(z)) + \frac{c}{12}S(w;z)\,, \tag{1.54}$$

where the so called Schwarzian derivative is introduced:

$$S(w;z) = \frac{(d^3w/dz^3)}{(dw/dz)} - \frac{3}{2}\left(\frac{(d^2w/dz^2)}{(dw/dz)}\right)^2\,. \tag{1.55}$$

The energy-momentum tensor is an example of a field that is quasi-primary but not primary. The Schwarzian derivative is, in fact, a unique weight two object that vanishes when restricted to the global $SL(2,C)$ subgroup of 2D conformal group. It satisfies the following composition law:

$$S(w,z) = \left(\frac{df}{dz}\right)^2 S(w,f) + S(f,z)\,. \tag{1.56}$$

For the exponential map $w \to z = e^w$, which maps the cylinder to the plain, one has

$$S(e^w, w) = -1/2\,, \tag{1.57}$$

therefore (1.54) will give

$$T_{\rm cyl}(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12} S(z,w) = z^2 T(z) - \frac{c}{24}\,. \tag{1.58}$$

Inserting mode expansion $T(z) = \sum L_n z^{-n-2}$, one obtains

$$T_{\rm cyl}(w) = \sum L_n z^{-n} - \frac{c}{24} = \sum_n \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{-nw}\,. \tag{1.59}$$

In particular the translation generator $(L_0)_{\rm cyl}$ on a cylinder is then given in terms of the generator L_0 on plane as:

$$(L_0)_{\rm cyl} = L_0 - \frac{c}{24}\,. \tag{1.60}$$

The central charge is seen to be proportional to the Casimir energy, the change in the vacuum energy density due to the finite circumference of the cylinder.

Virasoro algebra

As we saw in (1.27) the current of conformal transformations is $J(z) = T(z)\epsilon(z)$. Since $\epsilon(z)$ is an arbitrary holomorphic function, it is natural to expand it in its modes. We expect that the current $T(z)z^{n+1}$ generates the transformation $z \to z + c_n z^{n+1}$. According to (1.28) the corresponding charges are:

$$L_n = \frac{1}{2\pi i}\oint dz T(z) z^{n+1}. \tag{1.61}$$

The commutator of the charges is

$$[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 dw w^{m+1}\oint_w dz z^{n+1} T(z)T(w) = \frac{1}{12} cn(n^2-1)\delta_{n+m,0} + (n-m)L_{m+n}\,. \tag{1.62}$$

The classical generators of the local conformal transformations obey the Witt algebra (1.14). The quantum generators L_n obey an identical algebra, except for a central term:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{1}{12} c\, n(n^2-1)\delta_{n+m,0}\,; \tag{1.63}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{1}{12}\bar{c}\, n(n^2-1)\delta_{n+m,0}\,; \tag{1.64}$$

$$[L_n, \bar{L}_m] = 0\,. \tag{1.65}$$

The central extension of the Witt algebra is known as the Virasoro algebra. One can derive the last commutation relation (1.65) similarly by applying the fact that $T(z)$ and $\bar{T}(\bar{z})$ have no singularity in their OPE. Note that for $n = 0\,, \pm 1$ the central extension term vanishes and the quantum version of the global conformal group is still $SL(2,\mathbb{C})/\mathbb{Z}_2$.

Highest weight states

The vacuum state $|0\rangle$ should be invariant under global conformal transformations. This means that it must be annihilated by L_0 and $L_{\pm 1}$ and their antiholomorphic counterparts. Combining this with (1.40) we get

$$L_0|0\rangle = 0 \quad \text{and} \quad L_n|0\rangle = 0\,,\ \text{for all } n \geq -1\,. \tag{1.66}$$

It is reasonable to expect that the energy of our theory is bounded from below. Since $L_0 + \bar{L}_0$ is the Hamiltonian, we will assume that our representation contains a state with smallest value of L_0. This state is called the highest weight state. The highest weight states are related to the primary fields. More precisely every primary filed gives rise to a highest weight state. To see this let us consider a primary field $\Phi(z,\bar{z})$ of dimensions $(h,\bar{h})$. From the OPE (1.36) between $T(z)$ and the primary field $\Phi(z,\bar{z})$ one finds:

$$[L_n, \phi(w,\bar{w})] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w,\bar{w}) = h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial_w\phi(w,\bar{w})\,, \tag{1.67}$$

Inserting (1.38) for the modes of $\Phi(w)$ we get

$$[L_n, \Phi_m] = (n(h-1)-m)\,\Phi_{n+m}\,. \tag{1.68}$$

A spacial case of which is

$$[L_0, \Phi_m] = -m\Phi_m\,. \tag{1.69}$$

Thus, it seems reasonable to define

$$|h, \bar{h}\rangle = \Phi(0,0)|0\rangle\,. \tag{1.70}$$

Applying (1.67) to this state we see that

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle\,; \qquad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle\,. \tag{1.71}$$

Again from (1.67) we get

$$[L_n, \Phi(0,0)] = 0 \quad \text{for all} \quad n > 0\,. \tag{1.72}$$

Thus it is obvious that

$$L_n|h, \bar{h}\rangle = 0, \quad \bar{L}_n|h, \bar{h}\rangle = 0 \qquad \text{for all} \quad n > 0\,. \tag{1.73}$$

A state satisfying (1.71) and (1.73) is called a highest weight state. It follows from (1.63) that the negative modes L_n, $n < 0$, can be used to generate other states with larger dimensions:

$$L_0 L_n|h, \bar{h}\rangle = \left([L_0, L_n] + L_n L_0\right)|h, \bar{h}\rangle = (h - n)\, L_n|h, \bar{h}\rangle, \text{ with } n < 0\,. \tag{1.74}$$

This means that excited states may be obtained by successive applications of these operators on the highest weight state:

$$L_{-k_1} L_{-k_2} \ldots L_{-k_n}|h\rangle\,, \text{ where } \sum_{i=1}^{n} k_i = N\,. \tag{1.75}$$

We may in fact assume that the generators are ordered as: $1 \leq k_1 \leq \ldots \leq k_n$, since any incorrectly ordered product could be reduced to the ordered ones with the help of the Virasoro algebra commutation relations (1.63). The state (1.75) is an eigenstate of L_0 with the eigenvalue $h + N$. They are called descendants of the highest weight state $|h\rangle$. The collection of states (1.75) for all $n \geq 0$ could be ordered as:

level	dimension	state	# of states
0	h	$\|h\rangle$	1
1	$h+1$	$L_{-1}\|h\rangle$	1
2	$h+2$	$L_{-1}^2\|h\rangle,\ L_{-2}\|h\rangle$	2
3	$h+3$	$L_{-3}\|h\rangle,\ L_{-1}L_{-2}\|h\rangle,\ L_{-1}^3\|h\rangle$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$h+N$	$\ldots$	$P(N)$

The linear span of these states constitute the so called Verma module $V(c,h)$ of $|h\rangle$. In the table above we have denoted by $P(N)$ the number of partitions of N into positive integer parts. It is not difficult to see that

$$\sum_{n=0}^{\infty} P(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}, \qquad P(0) = 1\,. \tag{1.76}$$

In a similar manner we could construct a Verma module $\bar{V}(c,\bar{h})$ also with the antiholomorphic generators $\bar{L}_n$. In general the Hilbert space of a CFT is the direct sum of $V \otimes \bar{V}$, over the set of all conformal dimensions of primary states:

$$\sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h})\,. \tag{1.77}$$

Chapter 2

RG domain wall for the N=1 minimal superconformal models

2.1 Minimal models

The simplest of all conformal theories are the so called minimal models. In these theories the number of conformal families is finite. A well known example of a theory described by a unitary minimal model is the Ising model. Though in QFTs, unitarity is a fundamental requirement, in statistical mechanical systems it does not play such a central a role. Nevertheless in what follows we will restrict our attention to the unitary theories.

2.2 Unitary CFTs

In this section we will investigate the values of c and h for which the Virasoro algebra has unitary representations. By definition a representation of the Virasoro algebra is said to be unitary, if it does not contain negative-norm states.

Let us consider the norm of the state $L_{-n}|h\rangle$:

$$\langle h|L_nL_{-n}|h\rangle = \langle h|\left[L_nL_{-n}\right]|h\rangle = \left(2nh + \frac{1}{12}cn\left(n^2-1\right)\right)\langle h|h\rangle\,, \tag{2.1}$$

where in the last step we applied (1.63) and (1.71). Unitarity requires (2.1) to be positive for all $n > 0$. Thus:

- when $n = 1$, the conformal weight must be positive i.e. $h > 0$,
- when $n > 1$, the central charge must be positive i.e. $c > 0$.

We conclude that for unitary theories $h > 0$, and $c > 0$.

Null states and the Gram matrix

A descendant state is called a null state (or a singular vector) if it is a highest-weight state as well. Thus $|\chi\rangle$ is a null state if

$$L_0|\chi\rangle = (h+N)|\chi\rangle\,, \quad L_n|\chi\rangle = 0 \;\text{ for all } n > 0\,. \tag{2.2}$$

Singular vectors are orthogonal to the entire Verma module (this can be seen with the help of the Virasoro algebra relations (1.63) and the definitions of the highest weight states and the null states). As a consequence all descendants of a singular vector have zero norm too. To find the null states, and to find necessary and sufficient conditions for the unitarity it is helpful to consider the so called Gram matrix (denoted by M) of inner products between all basis states. Let us introduce some notations

$$|i\rangle \equiv L_{-k_1}L_{-k_2}\ldots L_{-k_n}|h\rangle\,, \quad M_{ij} = \langle i|j\rangle\,. \tag{2.3}$$

Note that the Gram matrix is Hermitian ($M^\dagger = M$) and it is block diagonal with blocks $M^{(l)}$ corresponding to states of level l. Then the norm of a generic state $|a\rangle = \sum_i a_i|i\rangle$ is $\langle a|a\rangle = a^\dagger M a$. Since M is Hermitian it can be diagonalized by a unitary matrix U:

$$M = U^\dagger \Lambda U, \quad \text{hence} \quad \langle a|a\rangle = \sum_i \Lambda_i |b_i|^2\,, \quad \text{where} \quad b \equiv Ua\,. \tag{2.4}$$

where the eigenvalues Λ_i are real numbers.

Let us calculate the matrices $M^{(l)}$ for the cases $l = 0, 1, 2$:

- $l = 0$, we have $M^{(0)} = \langle h|h\rangle = 1$;
- $l = 1$, we have $M^{(1)} = \langle h|L_1 L_{-1}|h\rangle = \langle h|[L_1, L_{-1}]|h\rangle = 2h$;
- $l = 2$, we have two descendants $L_{-1}^2|h\rangle$ and $L_{-2}|h\rangle$, thus

$$M^{(2)} = \begin{pmatrix} \langle h|L_1^2 L_{-1}^2|h\rangle, & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle, & \langle h|L_2 L_{-2}|h\rangle \end{pmatrix} = \begin{pmatrix} 4h(2h+1), & 6h \\ 6h, & 4h + c/2 \end{pmatrix} \tag{2.5}$$

We get no additional information from $M^{(0)}$. $M^{(1)}$ is a special case of (2.1) so we get no additional information again. As we know the determinant is equal to the product of its eigenvalues, in particular if one of the eigenvalues of $M^{(2)}$ is negative and the remaining eigenvalues are positive then $\det M^{(2)}$ is negative. Thus the negativity of the determinant indicates that the theory is not unitary. Explicitly

$$\det M^{(2)} = 32h^3 - 20h^2 + 4h^2 c + 2hc = 32\,(h - h_{1,1})\,(h - h_{1,2})\,(h - h_{2,1}) \ , \tag{2.6}$$

where

$$h_{1,1} = 0\,; \tag{2.7}$$
$$h_{1,2} = \tfrac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right) \ ; \quad h_{1,2} = \tfrac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right)$$

Another useful indicator is the trace of the Gram matrix which is equal to the sum of its eigenvalues:

$$\mathrm{tr} M^{(2)} = 8h(h+1) + c/2\,. \tag{2.8}$$

whenever the $\det M^{(2)}$ or $\mathrm{tr} M^{(2)}$ is negative, we conclude that the representation is not unitary.

2.2.1 Kac determinant

The generalization of (2.6) is

$$\det M^{(l)} = \alpha_l \prod_{rs\leq l} \left(h - h_{r,s}(c)\right)^{P(l-rs)} \,, \tag{2.9}$$

this formula is due to Kac and is called the Kac determinant. Where $P(l-rs)$ is the number of partitions of the integer $l-rs$ and α_l is a positive constant independent of h and c,

$$\alpha_l = \prod_{rs\leq l} \left((2r)^s s!\right)^{m(r,s)} \quad \text{where,} \quad m(r,s) = P(l-rs) - P(l-r(s+1))\,. \tag{2.10}$$

The products in (2.9) and (2.10) are over all positive integers r, s such that $r\,s \leq l$. The function $h_{r,s}(c)$ may be represented in various ways. Below we will give one of them that is convenient for our purposes.

$$h_{r,s}(p) = \frac{\left((p+1)r - ps\right)^2 - 1}{4p(p+1)} \,, \tag{2.11}$$

where the the central charge c is parametrized in terms of (in general complex) quantity p:

$$c_p = 1 - \frac{6}{p(p+1)}. \tag{2.12}$$

It is easy to check that (2.9) coincides with (2.6) when $l = 2$. Let us point out that the values of $h_{r,s}(p)$ in (2.10) do not change under replacement $r \to p - r$, $s \to p + 1 - s$.

To summarize if at any given level the Kac determinant is negative then there exist are negative norm states and the representation is not unitary. Instead if the Kac determinant is positive or equal to zero, then more subtle analysis is required to determine whether or not the representation is unitary at that level.

It can be proven that for the region $c \leq 1$ and $h \geq 0$, the necessary and sufficient conditions for a representations to be unitary are:

a) The central charge assumes one of the following values:

$$c_p = 1 - \frac{6}{p(p+1)}, \quad \text{where} \quad p = 3, 4, \dots . \tag{2.13}$$

Note that $p = 2$ just means $c = 0$.

b) To each c_p there are $p(p-1)/2$ primary fields, with conformal dimension:

$$h_{n,m} = \frac{((p+1)n - pm)^2 - 1}{4p(p+1)}, \tag{2.14}$$

where two integers take values $n \in \{1, 2, \cdots, p-1\}$, $m \in \{1, 2, \cdots, p\}$. The corresponding primary fields will be denoted as $\phi_{n,m}$.

2.2.2 N=1 minimal superconformal field theory

In any conformal field theory the energy-momentum tensor has two nonzero components: the holomorphic field $T(z)$ with conformal dimension $(2, 0)$ and its anti-holomorphic counterpart $\bar{T}(\bar{z})$ with dimensions $(0, 2)$. In $N = 1$ superconformal field theories one has in addition superconformal currents $G(z)$ and $\bar{G}(\bar{z})$ with dimensions $(3/2, 0)$ and $(0, 3/2)$ respectively. These fields satisfy the OPE rules

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \cdots, \tag{2.15}$$

$$T(z)G(0) = \frac{3G(0)}{2z^2} + \frac{G'(0)}{z} + \cdots, \tag{2.16}$$

$$G(z)G(0) = \frac{2c}{3z^3} + \frac{2T(0)}{z} + \cdots. \tag{2.17}$$

The corresponding expressions for the anti-chiral fields look exactly the same. One should simply substitute z by $\bar{z}$. Further on we'll mainly concentrate on the holomorphic part assuming similar expressions for anti-holomorphic quantities implicitly.

Due to the fermionic nature of the super current, there are two distinct possibilities for its behavior under the rotation of the argument around 0 by the angle 2π

$$G(e^{2\pi i}z) \quad = G(z) \qquad \textit{Neveu - Schwarz sector (NS)}\,, \tag{2.18}$$

$$G(e^{2\pi i}z) \qquad = -G(z) \qquad \textit{Ramond sector (R)}\,. \tag{2.19}$$

The space of fields $\mathcal{A}$ of the superconformal theory decomposes into a direct sum

$$\mathcal{A} = \{NS\} \oplus \{R\}\,, \tag{2.20}$$

where the subspaces $\{NS\}$ and $\{R\}$ consist of the Neveu-Shwarz and the Ramond fields respectively. By definition, the monodromy of $G(z)$ around a Neveu-Schwarz field is trivial (the case of eq. (2.18)) and its monodromy around a Ramond field produces a minus sign (the case of eq. (2.19)). Because of these two possibilities the Laurent expansions for the super-current will be

$$\begin{aligned} G(z) &= \sum_{k\in Z+1/2} \frac{G_k}{z^{k+3/2}} && \textit{Neveu-Schwarz sector (NS)}\,, \\ G(z) &= \sum_{k\in Z} \frac{G_k}{z^{k+3/2}} && \textit{Ramond sector (R)}\,. \end{aligned}$$

The OPE's (2.15), (2.16), (2.17) are equivalent to the Neveu-Schwarz-Ramond algebra relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\,, \\ [L_n, G_k] &= \frac{1}{2}(n-2k)G_{n+k}\,, \\ \{G_k, G_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2 - 1/4\right)\delta_{k+l,0}\,, \end{aligned} \tag{2.21}$$

where $\{,\}$ denotes the anticommutator. In this paper we'll deal with minimal super-conformal series denoted as SM_p $(p = 3, 4, 5 \ldots)$ corresponding to the choice of the central charge

$$c_p = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right)\,. \tag{2.22}$$

The main distinctive mark of the minimal super-conformal theories is that they have finitely many super primary fields. These fields are numerated by two integers $n \in \{1, 2, \cdots, p-1\}$, $m \in \{1, 2, \cdots, p+1\}$ and will be denoted as $\phi_{n,m}$. It is assumed that $\phi_{p-n,p+2-m} \equiv \phi_{n,m}$, hence the number of super primaries is equal to $[p^2/2]$ ([x] is the integer part of x). $\phi_{p-1,p+1} \equiv \phi_{1,1}$ is the identity operator. For even (odd) $n-m$ the super-conformal classes $[\phi_{n,m}]$ form irreducible representations of the Neveu-Schwarz (Ramond) algebra. The fields $\phi_{n,m}$ have dimensions

$$h_{n,m} = \frac{((p+2)n - pm)^2 - 4}{8p(p+2)} + \frac{1}{32}(1 - (-)^{n-m}) . \tag{2.23}$$

2.2.3 Current algebra and the coset construction

We will use the coset construction [23, 24] of super-minimal models in terms of $\widehat{SU}(2)_k$ WZNW models [25, 26].

Recall that WZNW models are endowed with spin one holomorphic currents. The OPE relations of these currents specified to the case of $\widehat{SU}(2)_k$ read:

$$\begin{aligned} J^0(z)J^0(0) &= \frac{k/2}{z^2} + reg\,, \\ J^0(z)J^{\pm}(0) &= \pm\frac{J^{\pm}(0)}{z} + reg\,, \\ J^+(z)J^-(0) &= \frac{k}{z^2} + \frac{2J^0(0)}{z} + reg\,, \end{aligned} \tag{2.24}$$

where k is the level. The isotopic indices $\pm, 0$ convenient for the later use are related to the usual Euclidean indices as:

$$J^0 \equiv J^3 \qquad and \qquad J^{\pm} \equiv J^1 \pm iJ^2 . \tag{2.25}$$

The Laurent expansion of the currents reads

$$J^a(z) = \sum_{n \in Z} \frac{J^a_n}{z^{n+1}} \tag{2.26}$$

and the OPE rules (2.24) imply that the current algebra generators are subject to the $Ka\check{c} - Moody$ algebra commutation relations

$$\begin{aligned}
\left[J^{\pm}_n, J^{\pm}_m\right] &= 0\,, \\
\left[J^{+}_n, J^{-}_m\right] &= kn\delta_{n+m,0} + 2J^0_{n+m}\,, \\
\left[J^{0}_n, J^{\pm}_m\right] &= \pm J^{\pm}_{n+m}\,, \\
\left[J^{0}_n, J^{0}_m\right] &= \frac{kn}{2}\delta_{n+m,0}\,.
\end{aligned} \tag{2.27}$$

Notice that the subalgebra generated by J^a_0 is simply the Lie algebra $su(2)$.

The energy momentum tensor can be expressed through the currents with the help of the Sugawara construction

$$T(z) = \frac{1}{k+2}\left(J^0J^0 + \frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+\right)\,. \tag{2.28}$$

As it is custom in CFT above and in what follows we assume that any product of local fields taken at coinciding points is regularised subtracting singular parts of the respective OPE. The central charge of the Virasoro algebra can be easily computed using (2.28). The result is:

$$c_k = \frac{3k}{k+2}\,. \tag{2.29}$$

The primary fields of the theory $\phi_{j,m}$ and corresponding states $|j,m\rangle$ are labeled by the spin of the representation $j = 0, 1/2, 1, \ldots, k/2$ and its projection $m = -j, -j+1, \ldots, j$. The corresponding conformal dimensions are given by

$$h = \frac{j(j+1)}{k+2}\,. \tag{2.30}$$

The zero modes of the currents act on the states $|j,m\rangle$ as *

$$\begin{aligned} J^{\pm}|j,m\rangle &= \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1\rangle\,, \\ J^{0}|j,m\rangle &= m|j,m\rangle\,. \end{aligned} \tag{2.31}$$

We'll need also the explicit form of the $su(2)$ WZNW modular matrices

$$S^{(k)}_{n,m} = \sqrt{\frac{2}{k+2}}\sin\frac{\pi nm}{k+2}\,. \tag{2.32}$$

It is well known that the $N=1$ super-minimal models can be represented as a coset [23, 24]

$$\mathcal{SM}_{k+2} = \frac{su(2)_k \times su(2)_2}{su(2)_{k+2}}\,. \tag{2.33}$$

In particular the energy momentum tensor of $\mathcal{SM}_{k+2}$ is given by

$$T_{(su(2)_k\times su(2)_2)/su(2)_{k+2}} = T_{su(2)_k} + T_{su(2)_2} - T_{su(2)_{k+2}}\,. \tag{2.34}$$

Indeed the combination of the central charges (2.29) corresponding to these three terms matches with the central charge of the super-minimal models (2.22).

The construction of the super-current G is more subtle; it involves the primary fields $\phi_{1,m}$ of the level $k=2$ WZNW theory (we denote the currents of this theory as K^a and summation over the index $a=\pm,0$ is assumed):

$$G(z) = C_a J^a(z)\phi_{1,-a}(z) + D_a K^a_{-1}\phi_{1,-a}(z)\,. \tag{2.35}$$

The coefficients C_a, D_a can be fixed requiring that the respective state be the highest weight state of the diagonal current algebra $J+K$. In other words both $J_0^+ + K_0^+$ and $J_1^+ + K_1^+$

*Note that a consistent with eq. (2.31) conjugation rule for the primary fields would be $\phi^{\dagger}_{j,m} = (-)^{j-m}\phi_{j,-m}$

annihilate the state

$$C_a J^a_{-1}|0\rangle|1,-a\rangle + D^a|0\rangle K^a_{-1}|1,-a\rangle\,. \tag{2.36}$$

Up to an overall constant κ we get

$$\begin{aligned}
&D_+ = \frac{\kappa}{\sqrt{2}}\,, \qquad && D_0 = \kappa\,, \qquad && D_- = -\frac{\kappa}{\sqrt{2}}\,,\\
&C_+ = -\frac{3\kappa\sqrt{2}}{k}\,, \qquad && C_0 = -\frac{6\kappa}{k}\,, \qquad && C_- = \frac{3\kappa\sqrt{2}}{k}\,.
\end{aligned} \tag{2.37}$$

The value of κ may be determined using the normalization condition of the the super-current fixed by the OPE (2.17)

$$\kappa = \sqrt{\frac{(k+2)(k+4)}{(k+6)(5k+54)}}\,, \tag{2.38}$$

but this won't be of importance for our goals.

2.2.4 Perturbative RG flows and domain walls

In a well known paper A. Zamolodchikov [11] has investigated the RG flow from minimal model $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ initiated by the relevant field $\phi_{1,3}$. Using leading order perturbation theory valid for $p >> 1$, for the several classes of local fields he calculated the mixing coefficients specifying the UV - IR map.

It was shown in [18] that a similar RG trajectory connecting $\mathcal{N}=1$ super-minimal models $\mathcal{SM}_p$ to $\mathcal{SM}_{p-2}$ exists. In this case the RG flow is initiated by the top component of the Neveu-Schwartz superfield $\Phi_{1,3}$. For us it will be important that also in this case a detailed analysis of some classes of fields has been carried out.

As it became clear later [19,22], above two examples are just the first simplest cases of more general RG flows. A wide class of CFT coset models

$$\mathcal{T}_{UV} \;=\; \frac{\hat{g}_l \times \hat{g}_m}{\hat{g}_{l+m}}\,, \qquad m > l \tag{2.39}$$

under perturbation by the relevant field $\phi = \phi_{1,1}^{Adj}$ [22] at the IR limit flow to the theories

$$\mathcal{T}_{IR} \;=\; \frac{\hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_m}\,. \tag{2.40}$$

Recently in [10] Gaiotto constructed a nontrivial conformal interface between successive minimal CFT models and made a striking proposal that this interface (RG domain wall) encodes the UV - IR map resulting through the RG flow discussed above. It was shown that the proposal agrees with the leading order perturbative analysis of [11].

Generalization of leading order calculations to a wider class of local fields [12] as well as next to leading order calculations [13, 14] further confirm the validity of this construction.

Actually in [10] Gaiotto suggests also a candidate for RG domain wall for the much more general RG flow between (2.39) and (2.40). Let us briefly recall the construction. Since a conformal interface between two CFT models is equivalent to some conformal boundary for the direct product of these theories (folding trick), it is natural to consider the product theory $\mathcal{T}_{UV} \times \mathcal{T}_{IR}$

$$\frac{\hat{g}_l \times \hat{g}_m}{\hat{g}_{m+l}} \times \frac{\hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_m} \sim \frac{\hat{g}_{m-l} \times \hat{g}_l \times \hat{g}_l}{\hat{g}_{l+m}}\,. \tag{2.41}$$

Notice the appearance of two identical factors $\hat{g}_l$ so one has a natural $\mathbb{Z}_2$ automorphism. Essentially the proposal of Gaiotto boils down to the statement that the boundary of the theory

$$\mathcal{T}_B = \frac{\hat{g}_l \times \hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_{l+m}}, \qquad m > l \tag{2.42}$$

acts as a $\mathbb{Z}_2$ twisting mirror. Explicitly the RG boundary condition is the image of the $\mathbb{Z}_2$

twisted $\mathcal{T}_B$ brane

$$|\tilde{B}\rangle = \sum_{s,t} \sqrt{S^{(m-l)}_{1,t} S^{(m+l)}_{1,s}} \sum_d |t, d, d, s; \mathcal{B}, Z_2\rangle\rangle, \tag{2.43}$$

where the indices t, d, s refer to the representations of $\hat{g}_{m-l}$, $\hat{g}_l$, $\hat{g}_{l+m}$ respectively and $S^{(k)}_{1,r}$ are the modular matrices of the $\hat{g}_k$ WZNW model.

In what follows we will examine in details the case of RG flow between $\mathcal{N} = 1$ super-minimal models. The method we apply directly explores the current algebra representation in contrary to the analysis in [10] where a specific representation applicable only for the unitary minimal series was used.

2.2.5 RG domain walls for super minimal models

In the case of the $\mathcal{N} = 1$ super-minimal models one should consider

$$\frac{\widehat{su}(2)_k \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} \times \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2}{\widehat{su}(2)_k} \sim \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2 \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}}\,, \tag{2.44}$$

where the first coset on lhs corresponds to the UV super conformal model $\mathcal{SM}_{k+2}$ and the second one to the IR theory $\mathcal{SM}_k$. We denote by $K(z)$ and $\widetilde{K}(z)$ the WZNW currents of $\widehat{su}(2)_2$ entering in the cosets of the IR and UV theories respectively. The current of $\widehat{su}(2)_{k-2}$ WZNW theory will be denoted as $J(z)$. Using (2.34) and the Sugawara construction, for the energy-momentum tensor of the IR theory (the second factor of the lhs of (2.44)) we get

$$T_{ir}(z) = \frac{1}{k} J(z)J(z) + \frac{1}{4} K(z)K(z) - \frac{1}{k+2} (K(z) + J(z))^2,$$

which can be rewritten as

$$T_{ir}(z) = \frac{2}{2k + k^2} J(z)J(z) - \frac{2}{2+k} J(z)K(z) + \frac{k-2}{4(k+2)} K(z)K(z). \tag{2.45}$$

Similarly the energy-momentum tensor for the UV theory is equal to

$$
\begin{aligned}
T_{uv}(z) = \frac{2}{(2+k)(4+k)} J(z)J(z) + \frac{2}{(2+k)(4+k)} K(z)K(z) \\
-\frac{2}{4+k} K(z)\widetilde{K}(z) + \frac{k}{4(k+4)} \widetilde{K}(z)\widetilde{K}(z) \\
+\frac{4}{(2+k)(4+k)} J(z)K(z) - \frac{2}{4+k} J(z)\widetilde{K}(z) \,.
\end{aligned}
\tag{2.46}
$$

In order to get the one-point functions of the theory $\mathcal{SM}_{k+2} \times \mathcal{SM}_k$ in the presence of RG boundary, one needs explicit expressions of the states corresponding to fields $\phi^{IR}\phi^{UV}$ in terms of the states of the coset theory

$$
\mathcal{T}_B = \frac{\widehat{su}(2)_{k-2} \times \widehat{su}(2)_2 \times \widehat{su}(2)_2}{\widehat{su}(2)_{k+2}} \,. \tag{2.47}
$$

Let us denote the highest weight representation spaces of the current algebras $J(z)$, $K(z)$ and $\widetilde{K}(z)$ as $V_j^{(J)}$, $V_k^{(K)}$ and $V_{\widetilde{k}}^{(\widetilde{K})}$ respectively (the lower indices specify the spins of the highest weight states). It is convenient to fix a unique representative of a state of the coset $\mathcal{T}_B$ in the space $V_j^{(J)} \otimes V_k^{(K)} \otimes V_{\widetilde{k}}^{(\widetilde{K})}$ requiring that the state under consideration be a highest weight state of the diagonal current $J + K + \widetilde{K}$. The simplest case to analyse are the states corresponding to $\phi^{IR}_{n,n}\phi^{UV}_{n,n}$. Since

$$
\begin{aligned}
h^{ir}_{n,n} &= \frac{n^2-1}{4k} - \frac{n^2-1}{4(k+2)} \,, \\
h^{uv}_{n,n} &= \frac{n^2-1}{4(k+2)} - \frac{n^2-1}{4(k+4)} \,,
\end{aligned}
$$

the total dimension of the product field is

$$
h^{ir}_{n,n} + h^{uv}_{n,n} = \frac{n^2-1}{4k} - \frac{n^2-1}{4(k+4)} \,, \tag{2.48}
$$

so that the corresponding state is readily identified with ($|j,m\rangle$ denotes a primary state of spin j and projection m)

$$|\frac{n-1}{2},\frac{n-1}{2}\rangle|0,0\rangle|0,0\rangle \in V^{(J)}_{\frac{n-1}{2}} \otimes V^{(K)}_0 \otimes V^{(\widetilde{K})}_0 \,. \tag{2.49}$$

Indeed, this is a spin $\frac{n-1}{2}$ highest weight state of the combined current $J+K+\widetilde{K}$ and its $\mathcal{T}_B$ dimension

$$h^{(J)}_{\frac{n-1}{2}} + h^{(K)}_0 + h^{(\widetilde{K})}_0 - h^{(J+K+\widetilde{K})}_{\frac{n-1}{2}}$$

coincides with (2.48). Notice that $\mathbb{Z}_2$ action (i.e. permutation of the second and third factors) on this state is trivial. Thus the overlap of this state with its $\mathbb{Z}_2$ image is equal to 1 and from (2.43)

$$\langle \phi^{IR}_{n,n}\phi^{UV}_{n,n}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}} \,. \tag{2.50}$$

For large k and for $n \sim O(1)$ this gives $1+3/k^2+O(1/k^3)$. We conclude that up to $1/k^2$ terms, the fields $\phi^{UV}_{n,n}$ flow to $\phi^{IR}_{n,n}$ without mixing with other fields, in complete agreement with both leading order [18] and next to leading order [21] perturbative calculations.

Next let us examine the more interesting case of Ramond fields $\phi^{UV}_{n,n\pm1}$ which are expected to flow to certain combinations of the fields $\phi^{IR}_{n\pm1,n}$ [18].

Consider the state corresponding to $\phi^{ir}_{n-1,n}\phi^{uv}_{n,n-1}$. From (2.23) we get

$$h^{ir}_{n-1,n} = \frac{3}{16} + \frac{(n-1)^2-1}{4k} - \frac{n^2-1}{4(k+2)} \,, \tag{2.51}$$

$$h^{uv}_{n,n-1} = \frac{3}{16} - \frac{(n-1)^2-1}{4(k+4)} + \frac{n^2-1}{4(k+2)} \,. \tag{2.52}$$

Hence the conformal dimension of this product field will be

$$h^{ir}_{n-1,n} + h^{uv}_{n,n-1} = \frac{3}{8} + \frac{(n-1)^2-1}{4k} - \frac{(n-1)^2-1}{4(k+4)} \,. \tag{2.53}$$

There are three primaries in $su(2)_2$ WZNW theory with $j = 0, 1, 2$ representations and conformal dimensions 0, $\frac{3}{16}$ and $\frac{1}{2}$ respectively. So, to get the right dimension one should choose a combination of states $|\frac{n}{2}-1, m\rangle|\frac{1}{2}, \alpha\rangle|\frac{1}{2}, \beta\rangle$. In addition this combination must be the spin $\frac{n}{2}-1$ highest weight state of $J + K + \widetilde{K}$ (to match with the last, negative term of (2.53)). Thus we are lead to

$$C_{\alpha\beta}|\frac{n}{2}-1, \frac{n}{2}-1-\alpha-\beta\rangle|\frac{1}{2}, \alpha\rangle|\frac{1}{2}, \beta\rangle, \tag{2.54}$$

where a summation over the indices $\alpha, \beta = \pm 1/2$ is assumed. The highest weight condition that the operator $J_0^+ + K_0^+ + \widetilde{K}_0$ annihilates this state, implies

$$\sqrt{n-2}C_{++} + C_{-+} + C_{+-} = 0\,.$$

A further constraint

$$C_{++} - \sqrt{n-2}C_{-+} = 0\,,$$

one obtains imposing the condition that this state should be an eigenstate of the Virasoro operator L_0^{IR} constructed from the energy-momentum tensor T_{ir} (2.45) with eigenvalue $h_{n,n-1}^{ir}$ (2.51). Thus we get

$$C_{++} = \sqrt{n-2}\,C_{-+}\,, \qquad C_{+-} = -(n-1)\,C_{-+}$$

(of course, the undefined overall multiplier could be fixed from the normalization condition). Taking (normalized) scalar product of the state (2.54) with its $\mathbb{Z}_2$ image we find

$$\langle\phi_{n-1,n}^{ir}\phi_{n,n-1}^{uv}|RG\rangle = -\frac{1}{n-1}\frac{\sqrt{S_{1,n-1}^{(k-2)}S_{1,n-1}^{(k+2)}}}{S_{1,n}^{k}}\,. \tag{2.55}$$

Consideration of the product $\phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}$ fields is quite similar and leads to the state

$$C_{\alpha\beta}|\frac{n}{2},\frac{n}{2}-\alpha-\beta\rangle|\frac{1}{2},\alpha\rangle|\frac{1}{2},\beta\rangle\,,$$

with the coefficients

$$C_{+-}=0\,,\qquad C_{++}=-\frac{1}{\sqrt{n}}C_{-+}\,.$$

So, in this case

$$\langle\phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}|RG\rangle=\frac{1}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n+1}S^{(k+2)}_{1,n+1}}}{S^{k}_{1,n}}\,. \tag{2.56}$$

Constructing the states corresponding to $\phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}$ and $\phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}$ is even simpler and one easily gets $|\frac{n}{2}-1,\frac{n}{2}-1\rangle|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{n}{2},\frac{n}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle$ respectively. In both cases the $\mathbb{Z}_2$ action is trivial, hence

$$\langle\phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}|RG\rangle=\frac{\sqrt{S^{(k-2)}_{1,n-1}S^{(k+2)}_{1,n+1}}}{S^{k}_{1,n}}\,, \tag{2.57}$$

$$\langle\phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}|RG\rangle=\frac{\sqrt{S^{(k-2)}_{1,n+1}S^{(k+2)}_{1,n-1}}}{S^{k}_{1,n}}\,. \tag{2.58}$$

In the large k limit we get

$$\langle\phi^{ir}_{n+1,n}\phi^{uv}_{n,n+1}|RG\rangle = \frac{1}{n}+O(1/k^2)\,, \tag{2.59}$$

$$\langle\phi^{ir}_{n+1,n}\phi^{uv}_{n,n-1}|RG\rangle = \frac{\sqrt{n^2-1}}{n}+O(1/k^2)\,, \tag{2.60}$$

$$\langle\phi^{ir}_{n-1,n}\phi^{uv}_{n,n+1}|RG\rangle = \frac{\sqrt{n^2-1}}{n}+O(1/k^2)\,, \tag{2.61}$$

$$\langle\phi^{ir}_{n-1,n}\phi^{uv}_{n,n-1}|RG\rangle = -\frac{1}{n}+O(1/k^2)\,, \tag{2.62}$$

in complete agreement with the second order perturbation theory results [21].

We have analysed also the more complicated case of mixing of the primary Neveu-Schwartz superfields $\Phi_{n,n\pm 2}$ and the descendant superfield $\mathbf{D}\bar{\mathbf{D}}\Phi_{n,n}$ (here $\mathbf{D}$ and $\bar{\mathbf{D}}$ are the super-derivatives). The details of calculations are presented in the appendix. Here are the final results:

$$\langle\psi^{ir}_{n+2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{2}{(n+1)(n+2)}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{2.63}$$

$$\langle\phi^{ir}_{n+2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{2.64}$$

$$\langle\psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n+2}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,, \tag{2.65}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n+1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{2.66}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{n^2-5}{n^2-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{2.67}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n}S^{(k+2)}_{1,n-2}}}{S^{(k)}_{1,n}}\,, \tag{2.68}$$

$$\langle\psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n+2}}}{S^{(k)}_{1,n}}\,, \tag{2.69}$$

$$\langle\phi^{ir}_{n-2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = -\frac{2}{n-1}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n}}}{S^{(k)}_{1,n}}\,, \tag{2.70}$$

$$\langle\phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}|RG\rangle = \frac{2}{(n-1)(n-2)}\frac{\sqrt{S^{(k-2)}_{1,n-2}S^{(k+2)}_{1,n-2}}}{S^{k}_{1,n}}\,. \tag{2.71}$$

At the large k limit we get

$$\langle\psi^{ir}_{n+2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n(n+1)} + O(1/k^2)\,, \tag{2.72}$$

$$\langle\phi^{ir}_{n+2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{2}{n+1}\sqrt{\frac{n+2}{n}} + O(1/k^2)\,, \tag{2.73}$$

$$\langle\psi^{ir}_{n+2,n}\psi^{uv}_{n,n-2}|RG\rangle = \frac{\sqrt{n^2-4}}{n} + O(1/k^2)\,, \tag{2.74}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n+2}|RG\rangle = \frac{2}{n+1}\sqrt{\frac{n+2}{n}} + O(1/k^2)\,, \tag{2.75}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = \frac{n^2-5}{n^2-1} + O(1/k^2)\,, \tag{2.76}$$

$$\langle G^{ir}_{-\frac{1}{2}}\phi^{ir}_{n,n}\phi^{uv}_{n,n-2}|RG\rangle = -\frac{2}{n-1}\sqrt{\frac{n-2}{n}} + O(1/k^2)\,, \tag{2.77}$$

$$\langle \psi^{ir}_{n-2,n}\psi^{uv}_{n,n+2}|RG\rangle = \frac{\sqrt{n^2-4}}{n} + O(1/k^2)\,, \tag{2.78}$$

$$\langle \phi^{ir}_{n-2,n}G^{uv}_{-\frac{1}{2}}\phi^{uv}_{n,n}|RG\rangle = -\frac{2}{n-1}\sqrt{\frac{n-2}{n}} + O(1/k^2)\,, \tag{2.79}$$

$$\langle \phi^{ir}_{n-2,n}\phi^{uv}_{n,n-2}|RG\rangle = \frac{2}{n(n-1)} + O(1/k^2)\,. \tag{2.80}$$

Again, the results are in complete agreement with the next to leading order perturbative calculations of [21].

It is interesting to note that, though the mixing coefficients computed here in the large k limit coincide with the respective cases of the $\phi_{1,3}$ perturbed minimal models, the exact k dependence in supersymmetric case enters solely through the modular matrices, in contrary to the quite complicated k dependence of the non supersymmetric case.

Chapter 3

Classical and semiclassical properties of the Liouville theory with defects

3.1 Classical Liouville theory with defects

3.1.1 Review of Liouville solution

Let us recall some facts on classical Liouville theory.

The action of the Liouville theory is

$$S = \frac{1}{2\pi i}\int \left(\partial\phi\bar{\partial}\phi + \mu\pi e^{2b\phi}\right) d^2z\,. \tag{3.1}$$

Here we use a complex coordinate $z = \tau + i\sigma$, and $d^2z \equiv dz \wedge d\bar{z}$ is volume form.

The field $\phi(z,\bar{z})$ satisfies the classical Liouville equation of motion

$$\partial\bar{\partial}\phi = \pi\mu b e^{2b\phi} \tag{3.2}$$

The general solution to (3.2) was given by Liouville in terms of two arbitrary functions $A(z)$ and $B(\bar{z})$ [46]

$$\phi = \frac{1}{2b}\ln\left(\frac{1}{\pi\mu b^2}\frac{\partial A(z)\bar{\partial}B(\bar{z})}{(A(z)+B(\bar{z}))^2}\right) \tag{3.3}$$

The solution (3.3) is invariant if one transforms A and B simultaneously by the following constant Möbius transformations:

$$A \to \frac{\alpha A + \beta}{\gamma A + \delta}, \qquad B \to \frac{\alpha B - \beta}{-\gamma B + \delta}, \qquad \alpha\delta - \beta\gamma = 1 \tag{3.4}$$

Classical expressions for left and right components of the energy-momentum tensor are

$$T = -(\partial\phi)^2 + b^{-1}\partial^2\phi \tag{3.5}$$

$$\bar{T} = -(\bar{\partial}\phi)^2 + b^{-1}\bar{\partial}^2\phi \tag{3.6}$$

Substituting (3.3) in (3.5) and (3.6) we get, that components of the energy-momentum tensor are given by the Schwarzian derivatives of $A(z)$ and $B(\bar{z})$:

$$T = \{A; z\} = \frac{1}{2b^2}\left[\frac{A'''}{A'} - \frac{3}{2}\frac{(A'')^2}{(A')^2}\right] \tag{3.7}$$

$$\bar{T} = \{B; \bar{z}\} = \frac{1}{2b^2}\left[\frac{B'''}{B'} - \frac{3}{2}\frac{(B'')^2}{(B')^2}\right] \tag{3.8}$$

The Schwarzian derivative is invariant under arbitrary constant Möbius transformation:

$$\left\{\frac{\alpha F + \beta}{\gamma F + \delta}; z\right\} = \{F; z\}, \qquad \alpha\delta - \beta\gamma = 1 \tag{3.9}$$

Solutions of the Liouville equation (3.2) can be described also via linear combination of some holomorphic and anti-holomorphic functions. Let us introduce the function $V = e^{-b\phi}$. One can write the Liouville equation (3.2) as equation for V

$$V\partial\bar{\partial}V - \partial V\bar{\partial}V = -\pi\mu b^2 \tag{3.10}$$

Also the left and right components of the energy-momentum tensor (3.5) and (3.6) can be wrtitten via V

$$\partial^2 V = -b^2 V T \tag{3.11}$$

$$\bar{\partial}^2 V = -b^2 V \bar{T} \tag{3.12}$$

It is straightforward to check that general solution of eq. (3.10) is given by linear combination of two holomorphic $a_i(z)$, $i = 1, 2$, and two anti-holomorphic $b_i(\bar{z})$, $i = 1, 2$ functions

$$V = \sqrt{\pi\mu b^2}\Big(a_1(z)b_1(\bar{z}) - a_2(z)b_2(\bar{z})\Big) \tag{3.13}$$

satisfying the condition

$$(a_1 a_2' - a_1' a_2)(b_1 b_2' - b_1' b_2) = 1 \tag{3.14}$$

Usually the fields $a_i(z)$ and $b_i(\bar{z})$, $i = 1, 2$ are normalized to have the unit Wronskian:

$$a_1 a_2' - a_1' a_2 = 1 \tag{3.15}$$

and

$$b_1 b_2' - b_1' b_2 = 1 \tag{3.16}$$

It is easy to see that the left and right components of the energy-momentum tensor can be expressed via a_i and b_i in the very simple form:

$$T = -\frac{1}{b^2}\frac{\partial^2 a_1}{a_1} = -\frac{1}{b^2}\frac{\partial^2 a_2}{a_2} \tag{3.17}$$

and

$$\bar{T} = -\frac{1}{b^2}\frac{\bar{\partial}^2 b_1}{b_1} = -\frac{1}{b^2}\frac{\bar{\partial}^2 b_2}{b_2} \tag{3.18}$$

The solutions (3.3) and (3.13) can be related in the following way. One can solve the unit Wronskian conditions (3.15) and (3.16) via holomorphic $A(z)$ and anti-holomorphic function

$B(\bar{z})$

$$a_1 = \frac{1}{\sqrt{\partial A}} \quad \text{and} \quad a_2 = \frac{A}{\sqrt{\partial A}} \tag{3.19}$$

and

$$b_1 = \frac{B}{\sqrt{\bar{\partial} B}} \quad \text{and} \quad b_2 = -\frac{1}{\sqrt{\bar{\partial} B}} \tag{3.20}$$

Inserting (3.19) and (3.20) in (3.13) we get (3.3). Note that Möbius transformations of A and B (3.4) become linear $SL(2,C)$ transformations of a_i and b_i:

$$\begin{aligned} \tilde{a}_1 &= \delta a_1 + \gamma a_2 \\ \tilde{a}_2 &= \beta a_1 + \alpha a_2 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \tilde{b}_1 &= \alpha b_1 + \beta b_2 \\ \tilde{b}_2 &= \gamma b_1 + \delta b_2 \end{aligned} \tag{3.22}$$

It is straightforward to check that indeed (3.13) is invariant under (3.21) and (3.22), and both of them keep the unit Wronskian condition.

One can also check, that both component of the energy-momentum tensor (3.17) and (3.18) are invariant under these transformations as well.

We finish this section with a remark which will be important in the parts devoted to light asymptotic limit. In that parts we will consider an analytic continuation $\mu \to -\mu$. At this point the solution (3.13) is convenient to write as:

$$V = \sqrt{-\pi\mu b^2}\Big(a_1(z)b_1(\bar{z}) + a_2(z)b_2(\bar{z})\Big) \tag{3.23}$$

It is easy to check (3.23) again solves the Liouville equation given that a_i and b_i, $i = 1, 2$ obey the condition (3.14).

3.1.2 Lagrangian of the Liouville theory with defect

Recently in [89] the action of the Liouville theory with topological defects was suggested:

$$S^{\text{top-def}} = \frac{1}{2\pi i}\int_{\Sigma_1}\left(\partial\phi_1\bar{\partial}\phi_1 + \mu\pi e^{2b\phi_1}\right)d^2z + \frac{1}{2\pi i}\int_{\Sigma_2}\left(\partial\phi_2\bar{\partial}\phi_2 + \mu\pi e^{2b\phi_2}\right)d^2z + \tag{3.24}$$
$$\int_{\partial\Sigma_1}\left[-\frac{1}{2\pi}\phi_2\partial_\tau\phi_1 + \frac{1}{2\pi}\Lambda\partial_\tau(\phi_1-\phi_2) + \frac{\mu}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b^2}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b - \kappa\right)\right]\frac{d\tau}{i}$$

Here Σ_1 is lower half-plane $\sigma = \text{Im}z \leq 0$, Σ_2 is upper half-plane $\sigma = \text{Im}z \geq 0$, and the defect is located along their common boundary, which is the real axis $\sigma = 0$ parametrized by $\tau = \text{Re}z$. Note that $\Lambda(\tau)$ here is additional field associated with the defect itself. The action (3.24) yields the following defect equations of motion at $\sigma = 0$:

$$\frac{1}{2\pi}(\partial-\bar{\partial})\phi_1 + \frac{1}{2\pi}\partial_\tau\phi_2 - \frac{1}{2\pi}\partial_\tau\Lambda + \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1-\phi_2)b = 0 \tag{3.25}$$

$$-\frac{1}{2\pi}(\partial-\bar{\partial})\phi_2 - \frac{1}{2\pi}\partial_\tau\phi_1 + \frac{1}{2\pi}\partial_\tau\Lambda + \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} + \frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1-\phi_2)b = 0 \tag{3.26}$$

$$\frac{1}{2\pi}\partial_\tau(\phi_1-\phi_2) - \frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b} - \frac{1}{\pi b}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b - \kappa\right) = 0 \tag{3.27}$$

The last equation is derived taking variation by Λ.

Using that $\partial_\tau = \partial + \bar{\partial}$ and forming various linear combinations of equations (3.25)-(3.27) we can bring them to the form:

$$\bar{\partial}(\phi_1-\phi_2) = \pi\mu b e^{b(\phi_1+\phi_2)}e^{-\Lambda b} \tag{3.28}$$

$$\partial(\phi_1-\phi_2) = \frac{2}{b}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b - \kappa\right) \tag{3.29}$$

$$\partial(\phi_1+\phi_2) - \partial_\tau\Lambda = \frac{2}{b}e^{\Lambda b}\sinh(b(\phi_1-\phi_2)) \tag{3.30}$$

It is shown in [89] that requiring that defect equations of motion hold for every σ brings additionally to the condition, that Λ is restriction to the real axis of a holomorphic field

$$\bar{\partial}\Lambda = 0 \tag{3.31}$$

This condition allows to rewrite (3.30) in the form

$$\partial(\phi_1 + \phi_2 - \Lambda) = \frac{2}{b} e^{\Lambda b} \sinh(b(\phi_1 - \phi_2)) \tag{3.32}$$

It is checked in [89] that the system of the defect equations of motion (3.28)-(3.32) guarantees that both components of the energy-momentum tensor are continuous across the defects and therefore describes topological defects:

$$-(\partial\phi_1)^2 + b^{-1}\partial^2\phi_1 = -(\partial\phi_2)^2 + b^{-1}\partial^2\phi_2 \tag{3.33}$$

$$-(\bar{\partial}\phi_1)^2 + b^{-1}\bar{\partial}^2\phi_1 = -(\bar{\partial}\phi_2)^2 + b^{-1}\bar{\partial}^2\phi_2 \tag{3.34}$$

Another interesting consequence of the defect equations of motion found in [89] is existence together with holomorphic field $\Lambda(z)$ an anti-holomorphic field Ξ:

$$\partial\Xi = 0 \tag{3.35}$$

where

$$\Xi = e^{-b(\phi_1+\phi_2)} e^{b\Lambda} (\cosh b(\phi_1 - \phi_2) - \kappa) \tag{3.36}$$

or alternatively

$$\Xi = \frac{b}{2} e^{-b(\phi_1+\phi_2)} \partial(\phi_1 - \phi_2) \tag{3.37}$$

Now we will present general solution of the defect equations of motion (3.28)-(3.32).

We will follow essentially the same strategy which was used in [47] for analyzing the boundary Liouville problem. On the one hand since the defect is topological both components of the energy-momentum tensor are equal being computed in terms of ϕ_1 or ϕ_2. On the other hand each component of the energy-momentum tensor is given by the Schwarzian derivative, which is invariant under the Möbius transformation. This naturally leads to the following Ansatz:

$$\phi_1 = \frac{1}{2b} \ln \left(\frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A+B)^2} \right) \tag{3.38}$$

$$\phi_2 = \frac{1}{2b} \ln \left(\frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} D}{(C+D)^2} \right) \tag{3.39}$$

where

$$C = \frac{\alpha A + \beta}{\gamma A + \delta} \quad \text{and} \quad D = \frac{\alpha' B + \beta'}{\gamma' B + \delta'} \tag{3.40}$$

Remembering that the ϕ_2 is invariant under the simultaneous Möbius transformation (3.4) of C and D, we can set $B = D$. Therefore without loosing generality we can consider an ansatz:

$$\phi_1 = \frac{1}{2b} \ln \left(\frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A+B)^2} \right) \tag{3.41}$$

$$\phi_2 = \frac{1}{2b} \ln \left(\frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} B}{(C+B)^2} \right) \tag{3.42}$$

where

$$C = \frac{\alpha A + \beta}{\gamma A + \delta} \tag{3.43}$$

Substituting (3.41) and (3.42) in (3.28) we find that it is satisfied with

$$e^{-\Lambda b} = \frac{A - C}{\sqrt{\partial A \partial C}} \tag{3.44}$$

Since A and C are holomorphic functions, Λ is holomorphic as well, as it is stated in (3.31).

It is straightforward to check that (3.32) is satisfied as well with ϕ_1, ϕ_2 and Λ given by (3.41), (3.42) and (3.44) respectively. And finally inserting (3.41), (3.42) and (3.44) in (3.29) we see that it is also fulfilled with

$$\kappa = \frac{\alpha + \delta}{2} \tag{3.45}$$

Inserting (3.41), (3.42) in (3.37) one can check that

$$\Xi = \frac{\pi\mu b^2}{4} \frac{\gamma B^2 + B(\alpha - \delta) - \beta}{\bar{\partial} B} \tag{3.46}$$

Remembering that B is anti-holomorphic we see that Ξ is anti-holomorphic as well.

We can write the solution of the defect equations of motion also using solutions of the Liouville equation in the form (3.13). Recalling that the Möbius transformation of functions A and B becomes linear $SL(2, C)$ transformation of functions a_i and b_i, which leaves the component of the energy-momentum tensor (3.17) and (3.18) invariant, we can write the ansatz in the form

$$e^{-b\phi_1} = \sqrt{\pi\mu b^2}\Big(a_1(z)b_1(\bar{z}) - a_2(z)b_2(\bar{z})\Big) \tag{3.47}$$

$$e^{-b\phi_2} = \sqrt{\pi\mu b^2}\Big(c_1(z)b_1(\bar{z}) - c_2(z)b_2(\bar{z})\Big) \tag{3.48}$$

where denoting $\vec{a} = (a_1, a_2)$, $\vec{c} = (c_1, c_2)$, and $D = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}$, one has

$$\vec{c} = D\vec{a} \tag{3.49}$$

and

$$2\kappa = \mathrm{Tr} D \tag{3.50}$$

3.1.3 Lagrangian of the Liouville theory with permutation branes

We can construct also folded version of the action (3.24) describing product of Liouville theories on half-plane with boundary condition given by permutation branes:

$$S^{\text{perm}-\text{brane}} = \frac{1}{2\pi i} \int_\Sigma \left(\partial\phi_1\bar{\partial}\phi_1 + \mu\pi e^{2b\phi_1} + \partial\phi_2\bar{\partial}\phi_2 + \mu\pi e^{2b\phi_2}\right) d^2 z + \tag{3.51}$$

$$\int_{\partial\Sigma}\left[-\frac{1}{2\pi}\phi_2\partial_\tau\phi_1+\frac{1}{2\pi}\Lambda\partial_\tau(\phi_1-\phi_2)-\frac{\mu}{2}e^{(\phi_1+\phi_2-\Lambda)b}+\frac{1}{\pi b^2}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b-\kappa\right)\right]\frac{d\tau}{i}$$

Σ denotes here upper half-plane $\sigma \geq 0$, and τ parameterizes boundary located at $\sigma = 0$. This action gives rise to boundary equations

$$\frac{1}{2\pi}(\partial-\bar{\partial})\phi_1+\frac{1}{2\pi}\partial_\tau\phi_2-\frac{1}{2\pi}\partial_\tau\Lambda-\frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b}+\frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1-\phi_2)b=0 \tag{3.52}$$

$$\frac{1}{2\pi}(\partial-\bar{\partial})\phi_2-\frac{1}{2\pi}\partial_\tau\phi_1+\frac{1}{2\pi}\partial_\tau\Lambda-\frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b}-\frac{1}{\pi b}e^{\Lambda b}\sinh(\phi_1-\phi_2)b=0 \tag{3.53}$$

$$\frac{1}{2\pi}\partial_\tau(\phi_1-\phi_2)+\frac{\mu b}{2}e^{(\phi_1+\phi_2-\Lambda)b}+\frac{1}{\pi b}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b-\kappa\right)=0 \tag{3.54}$$

Again using that $\partial_\tau = \partial + \bar{\partial}$ and taking various linear combinations, one can bring the system (3.52)-(3.54) to the form

$$\partial\phi_2-\bar{\partial}\phi_1=\pi\mu b e^{b(\phi_1+\phi_2)}e^{-\Lambda b} \tag{3.55}$$

$$\partial\phi_1-\bar{\partial}\phi_2=-\frac{2}{b}e^{\Lambda b}\left(\cosh(\phi_1-\phi_2)b-\kappa\right) \tag{3.56}$$

$$\partial\phi_1+\bar{\partial}\phi_2-\partial_\tau\Lambda=-\frac{2}{b}e^{\Lambda b}\sinh(b(\phi_1-\phi_2)) \tag{3.57}$$

One can check that equations (3.55)-(3.57) imply the permutation branes conditions:

$$T^{(1)}=\bar{T}^{(2)}|_{\sigma=0} \tag{3.58}$$
$$\bar{T}^{(1)}=T^{(2)}|_{\sigma=0}$$

or using (3.5) and (3.6)

$$-(\partial\phi_1)^2+b^{-1}\partial^2\phi_1=-(\bar{\partial}\phi_2)^2+b^{-1}\bar{\partial}^2\phi_2 \tag{3.59}$$

$$-(\bar{\partial}\phi_1)^2 + b^{-1}\bar{\partial}^2\phi_1 = -(\partial\phi_2)^2 + b^{-1}\partial^2\phi_2 \tag{3.60}$$

We can solve equations (3.55)-(3.57) using the same strategy, with only difference that now Möbius transformation relates holomorphic and antiholomorphic functions:

$$\phi_1 = \frac{1}{2b}\ln\left(\frac{1}{\pi\mu b^2}\frac{\partial A\bar{\partial}B}{(A+B)^2}\right) \tag{3.61}$$

$$\phi_2 = \frac{1}{2b}\ln\left(\frac{1}{\pi\mu b^2}\frac{\partial B\bar{\partial}C}{(C+B)^2}\right) \tag{3.62}$$

and

$$C = \frac{\alpha A + \beta}{\gamma A + \delta} \tag{3.63}$$

One can check that this ansatz satisfies the equation (3.55) with the Λ given by the relation

$$e^{-\Lambda b} = \frac{C - A}{\sqrt{\partial A\bar{\partial}C}} \tag{3.64}$$

It is straightforward to see that the ansatz (3.61)-(3.63) together with the Λ given by (3.64) solves also eq. (3.57).

And finally inserting ϕ_1, ϕ_2 and Λ given by (3.61), (3.62) and (3.64) respectively in eq. (3.56) one can check that it is satisfied as well with the following κ

$$\kappa = \frac{\alpha + \delta}{2} \tag{3.65}$$

3.2 Permutation branes and defects in Quantum Liouville

3.2.1 Review of quantum Liouville

Liouville field theory is conformal field theory with the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3-n)\delta_{n,-m} \tag{3.66}$$

with central charge

$$c_L = 1 + 6Q^2 \tag{3.67}$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q-\alpha) \tag{3.68}$$

The fields V_α and $V_{Q-\alpha}$ have the same conformal dimensions and represent the same primary field, i.e. they are proportional to each other:

$$V_\alpha = S(\alpha)V_{Q-\alpha} \tag{3.69}$$

with the reflection function

$$S(\alpha) = \frac{(\pi\mu\gamma(b^2))^{b^{-1}(Q-2\alpha)}}{b^2}\frac{\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))}{\Gamma(b(Q-2\alpha))\Gamma(1+b^{-1}(Q-2\alpha))} \tag{3.70}$$

Two-point functions of Liouville theory are given by the reflection function (3.70):

$$\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\rangle = \frac{S(\alpha)}{(z_1-z_2)^{2\Delta_\alpha}(\bar{z}_1-\bar{z}_2)^{2\Delta_\alpha}} \tag{3.71}$$

Introducing ZZ function [48]:

$$W(\alpha) = -\frac{2^{3/4} e^{3i\pi/2} (\pi\mu\gamma(b^2))^{-\frac{(Q-2\alpha)}{2b}} \pi(Q-2\alpha)}{\Gamma(1-b(Q-2\alpha))\Gamma(1-b^{-1}(Q-2\alpha))}\,. \tag{3.72}$$

two-point function can be compactly written as

$$S(\alpha) = \frac{W(Q-\alpha)}{W(\alpha)}\,, \tag{3.73}$$

Another useful property of ZZ function is

$$W(Q-\alpha)W(\alpha) = -i2\sqrt{2}\sin\pi b^{-1}(2\alpha-Q)\sin\pi b(2\alpha-Q)\,. \tag{3.74}$$

The spectrum of the Liouville theory has the form

$$\mathcal{H} = \int_0^\infty dP\; R_{\frac{Q}{2}+iP} \otimes R_{\frac{Q}{2}+iP} \tag{3.75}$$

where R_α is the highest weight representation with respect to the Virasoro algebra.

3.2.2 Permutation branes and defects in quantum Liouville

Let us recall the form of continuous family of defects and permutation branes in the Liouville field theory computed in [76, 82] using appropriate generalization of the Cardy-Lewellen equation [56].

Topological defects are intertwining operators X commuting with the Virasoro generators

$$[L_n, X] = [\bar{L}_n, X] = 0 \tag{3.76}$$

Such operators have the form

$$X = \int_{\frac{Q}{2}+i\mathbb{R}} d\alpha\, D(\alpha)\, \mathbb{P}^\alpha \tag{3.77}$$

where $\mathbb{P}^\alpha$ are projectors on a subspace $R_\alpha \otimes R_\alpha$:

$$\mathbb{P}^\alpha = \sum_{N,M} (|\alpha, N\rangle \otimes \overline{|\alpha, M\rangle})(\langle\alpha, N| \otimes \overline{\langle\alpha, M|}) \tag{3.78}$$

Here $|\alpha, N\rangle$ and $\overline{|\alpha, M\rangle}$ are vectors of orthonormal bases of left and right copy of R_α respectively. The eigenvalues $D(\alpha)$ can be determined via the two-point functions computed in the presence of defect X

$$\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2)\rangle = \frac{D(\alpha)S(\alpha)}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}} \tag{3.79}$$

It is shown in [82] that

$$\langle V_\alpha(z_1, \bar{z}_1) X_s V_\alpha(z_2, \bar{z}_2)\rangle = -\frac{1}{W^2(\alpha)} \frac{2^{1/2} i \cosh(2\pi s(2\alpha - Q))}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}} \tag{3.80}$$

and therefore for $D_s(\alpha)$ one can write using (3.73) and (3.74)

$$D_s(\alpha) = \frac{2^{1/2} \cosh(2\pi s(2\alpha - Q))}{S(\alpha) W^2(\alpha)} = \frac{\cosh(2\pi s(2\alpha - Q))}{2 \sin \pi b^{-1}(2\alpha - Q) \sin \pi b(2\alpha - Q)} \tag{3.81}$$

Parameter s is continuous parameter labeling a defect. Defects can be characterized also by the value of two-point function of a degenerate field $-b/2$ in the presence of defect. It is a function $A(b)$ of b. It is shown in [82] that parameter s related to the $A(b)$ by the equation:

$$2 \cosh 2\pi b s = A(b) \left(\frac{W(-b/2)}{W(0)} \right)^2 . \tag{3.82}$$

Permutation branes on product $L_1 \times L_2$ of two Liouville theories are given by gluing condition:

$$\begin{aligned} L_n^{(1)} - \bar{L}_{-n}^{(2)} &= 0, \\ L_n^{(2)} - \bar{L}_{-n}^{(1)} &= 0. \end{aligned} \tag{3.83}$$

Comparing gluing conditions (3.83) and (3.76) one can see that topological defects related to permutation branes by folding trick, consisting of exchanging left and right components of the second copy, and hence these branes are characterized by the same two-point functions (3.80) with z_2 and $\bar{z}_2$ exchanged

$$\langle V_\alpha^{(1)}(z_1,\bar{z}_1)V_\alpha^{(2)}(z_2,\bar{z}_2)\rangle_{\mathcal{P}} = -\frac{1}{W^2(\alpha)}\frac{2^{1/2}i\cosh(2\pi s(2\alpha-Q))}{(z_1-\bar{z}_2)^{2\Delta_\alpha}(\bar{z}_1-z_2)^{2\Delta_\alpha}} \tag{3.84}$$

3.3 Semiclassical limits

3.3.1 Heavy asymptotic limit

Let us consider the action (3.1) for the rescaled variable $\varphi = 2b\phi$

$$S = \frac{1}{8\pi i b^2}\int\left(\partial\varphi\bar{\partial}\varphi + 4\lambda e^{\varphi}\right)d^2z\,. \tag{3.85}$$

where $\lambda = \pi\mu b^2$.

This form shows that b^2 plays in the Liuoville theory the role of the Planck constant, and one can study semiclassical limit taking the limit $b\to 0$, in such a way that the value of the λ is kept fixed.

Now consider the correlation functions in the path integral formalism:

$$\left\langle V_{\alpha_1}(z_1,\bar{z}_1)\cdots V_{\alpha_n}(z_n,\bar{z}_n)\right\rangle = \int\mathcal{D}\varphi\; e^{-S}\prod_{i=1}^{n}\exp\left(\frac{\alpha_i\varphi(z_i,\bar{z}_i)}{b}\right) \tag{3.86}$$

We would like to calculate this integral in the semiclassical limit $b\to 0$ using the method of steepest descent, and we should decide how α_i scales with b. Since S scales b^{-2}, for operator to affect saddle point, we should take $\alpha_i = \eta_i/b$, with η_i fixed. The conformal weights $\Delta_\alpha = \eta(1-\eta)/b^2$ scale like b^{-2} as well. This is heavy asymptotic limit. Another choice of the operator scaling will be discussed in the next subsection.

We see from (3.86) that in the semiclasscial limit the correlation function is given by $e^{-S_{\rm cl}}$

where, at least naively, in a sense which will be clarified below, $S_{\rm cl}$ is the action

$$S = \frac{1}{8\pi i b^2} \int \left(\partial\varphi\bar{\partial}\varphi + 4\lambda e^{\varphi} \right) d^2z + \sum_{i=1}^{n} \frac{\eta_i}{b^2} \varphi(z_i, \bar{z}_i) \tag{3.87}$$

evaluated on the solution of its equation of motion:

$$\partial\bar{\partial}\varphi = 2\lambda e^{\varphi} - 4\pi \sum_{i=1}^{n} \eta_i \delta^2(z - z_i) \tag{3.88}$$

Assuming that in the vicinity of the insertion point z_i, one can ignore the exponential term we get that in the vicinity of the point z_i φ has the following behavior

$$\varphi(z, \bar{z}) = -4\eta_i \log |z - z_i| + X_i \quad \text{as} \quad z \to z_i \tag{3.89}$$

One can insert this solution back into the equation of motion to check, if indeed the exponential term is subleading. We find, that this happens when

$$\mathrm{Re}\eta_i < \frac{1}{2} \tag{3.90}$$

This constraint is known as Seiberg bound [94]. It is semiclassial version of the quantum condition (3.69) stating that V_α and $V_{Q-\alpha}$ represent the same quantum operator. Either α or $Q - \alpha$ always obey Seiberg bound.

Remembering that in the Liouville theory we have also background charge at infinity, conditions (3.89) should be complemented by the behavior at the infinity:

$$\varphi(z, \bar{z}) = -2 \log |z|^2 \quad \text{as} \quad |z| \to \infty \tag{3.91}$$

Since the energy-momentum tensor in the presence of the primary fields acquires quadratic singularity, functions a_i, $i = 1, 2$, should solve the equation

$$\partial^2 a_i + b^2 T a_i = 0 \tag{3.92}$$

where

$$b^2 T = \sum_{k=1}^{n} \frac{\eta_k(1-\eta_k)}{(z-z_k)^2} + \frac{c_k}{(z-z_k)} \tag{3.93}$$

where c_k are so called accessory parameters.

If one tries naively to evaluate the action (3.87) on a solution obeying (3.89), we find that it diverges. Therefore we should consider a regularized action. It was constructed in [95]:

$$\begin{aligned} b^2 S^{\text{reg}} &= \frac{1}{8\pi i} \int_{D-\cup_i d_i} \left(\partial\varphi\bar{\partial}\varphi + 4\lambda e^{\varphi} \right) d^2 z + \frac{1}{2\pi} \oint_{\partial D} \varphi d\theta + 2\log R \\ &- \sum_{i=1}^{n} \left(\frac{\eta_i}{2\pi} \oint_{\partial d_i} \varphi d\theta_i + 2\eta_i^2 \log \epsilon_i \right) \end{aligned} \tag{3.94}$$

Here D is a disc of radius R, d_i is a disc of radius ϵ_i around z_i. It was shown in [95] that the action (3.94) satisfies the equation

$$\frac{\partial}{\partial \eta_i} b^2 S^{\text{reg}} = -X_i \tag{3.95}$$

where X_i is defined by the boundary condition (3.89).

The Polyakov's conjecture proved in [49] states, that the action (3.94) also obeys the relation:

$$\frac{\partial}{\partial z_i} b^2 S^{\text{reg}} = -c_i \tag{3.96}$$

Let us write down regularized version of the action with defect.

First of all let us write it in the terms of $\lambda = \pi\mu b^2$, $\varphi_1 = 2b\phi_1$, $\varphi_2 = 2b\phi_2$, and $\tilde{\Lambda} = 2b\Lambda$:

$$\begin{aligned} b^2 S^{\text{top-def}} &= \frac{1}{8\pi i} \int_{\Sigma_1} \left(\partial\varphi_1\bar{\partial}\varphi_1 + 4\lambda e^{\varphi_1} \right) d^2 z + \frac{1}{8\pi i} \int_{\Sigma_2} \left(\partial\varphi_2\bar{\partial}\varphi_2 + 4\lambda e^{\varphi_2} \right) d^2 z + \\ &\int_{\partial\Sigma_1} \left[-\frac{1}{8\pi}\varphi_2\partial_\tau\varphi_1 + \frac{1}{8\pi}\tilde{\Lambda}\partial_\tau(\varphi_1-\varphi_2) + \frac{\lambda}{2\pi} e^{(\varphi_1+\varphi_2-\tilde{\Lambda})/2} - \frac{1}{\pi} e^{\tilde{\Lambda}/2} \left(\cosh\left(\frac{\varphi_1-\varphi_2}{2} \right) - \kappa \right) \right] \frac{d\tau}{i} \end{aligned} \tag{3.97}$$

Since we consider here only insertion of the bulk field, and do not consider insertion of the defect or boundary fields, the regularized action takes the form:

$$
\begin{aligned}
b^2 S^{\text{top-def}} &= \frac{1}{8\pi i}\int_{\Sigma_1^R - \cup_i d_i}\left(\partial\varphi_1\bar{\partial}\varphi_1 + 4\lambda e^{\varphi_1}\right)d^2z + \\
&- \sum_{i=1}^{n}\left(\frac{\eta_i}{2\pi}\oint_{\partial d_i}\varphi_1 d\theta_i + 2\eta_i^2\log\epsilon_i\right) + \frac{1}{2\pi}\int_{s_{R1}}\varphi_1 d\theta + \log R \\
&+ \frac{1}{8\pi i}\int_{\Sigma_2^R - \cup_j d_j}\left(\partial\varphi_2\bar{\partial}\varphi_2 + 4\lambda e^{\varphi_2}\right)d^2z + \\
&- \sum_{j=1}^{m}\left(\frac{\eta_j}{2\pi}\oint_{\partial d_j}\varphi_2 d\theta_j + 2\eta_j^2\log\epsilon_j\right) + \frac{1}{2\pi}\int_{s_{R2}}\varphi_2 d\theta + \log R \\
&+ \int_{\partial\Sigma_1}\left[-\frac{1}{8\pi}\varphi_2\partial_\tau\varphi_1 + \frac{1}{8\pi}\tilde{\Lambda}\partial_\tau(\varphi_1-\varphi_2) + \frac{\lambda}{2\pi}e^{(\varphi_1+\varphi_2-\tilde{\Lambda})/2} - \frac{1}{\pi}e^{\tilde{\Lambda}/2}\left(\cosh\left(\frac{\varphi_1-\varphi_2}{2}\right) - \kappa\right)\right]\frac{d\tau}{i}
\end{aligned}
\tag{3.98}
$$

where Σ_i^R is a half-disc of the radius R and s_{Ri} is a semicircle of the radius R in the half-plane Σ_i, $i = 1, 2$.

3.3.2 Light asymptotic limit

Another limit is so called light asymptotic limit. Here we take

$$\alpha = b\eta \tag{3.99}$$

In this limit the operator insertions have no influence and components of the energy-momentum tensor are (anti-) holomorphic and regular functions everywhere on sphere and thus vanish. Eq. (3.11) and (3.12) imply that $V \equiv e^{-b\phi}$ should be at the most first degree of z and $\bar{z}$, hence leading to the solutions * :

$$V(z,\bar{z};R) = \sqrt{-\lambda}(sz\bar{z} + tz + u\bar{z} + v)\,, \qquad R = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \tag{3.100}$$

where

$$\det R = sv - ut = 1 \tag{3.101}$$

*It is shown in [93] that to have solution in light limit one needs to perform analytical continuation $\mu \to -\mu$.

Thus the path integral in the light limit becomes finite-dimensional integral over parameters (s,t,u,v) which besides constraint (3.101) may satisfy some additional constraints like reality and defect/boundary condition. The reality of V requires the matrix R to be Hermitian. A way to parameterize hermitian matrices R is

$$R = \begin{pmatrix} X_0 - X_1 & X_2 + iX_3 \\ X_2 - iX_3 & X_0 + X_1 \end{pmatrix} \tag{3.102}$$

where $X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1$, makes clear that moduli space of solutions is three-dimensional hyperboloid H_3^+. Hence, fox example in the bulk Liouville theory, correlation function in the light asymptotic limit takes the form

$$\left\langle V_{b\eta_1}(z_1,\bar{z}_1)\cdots V_{b\eta_n}(z_n,\bar{z}_n)\right\rangle^{\text{light}} = \int_{H_3^+} dR \prod_{i=1}^{n} V^{-2\eta_i}(z_i,\bar{z}_i;R) \tag{3.103}$$

3.4 Defects in light asymptotic limit

Let us now specialize the light asymptotic limit rules to the defects. We should find solutions for ϕ_1 and ϕ_2 in the form (3.100) satisfying the defect equations of motion. One can check that expressions

$$V_1(z,\bar{z};R_1) = \sqrt{-\lambda}(s_1 z\bar{z} + t_1 z + u_1\bar{z} + v_1)\,, \quad R_1 = \begin{pmatrix} s_1 & t_1 \\ u_1 & v_1 \end{pmatrix}\,, \quad \det R_1 = 1 \tag{3.104}$$

$$V_2(z,\bar{z};R_2) = \sqrt{-\lambda}(s_2 z\bar{z} + t_2 z + u_2\bar{z} + v_2)\,, \quad R_2 = \begin{pmatrix} s_2 & t_2 \\ u_2 & v_2 \end{pmatrix}\,, \quad \det R_2 = 1 \tag{3.105}$$

satisfy the defect equations of motion (3.28)-(3.32) with

$$2\kappa = \mathrm{Tr}\left(R_2 R_1^{-1}\right) = s_1 v_2 + s_2 v_1 - u_1 t_2 - u_2 t_1 \tag{3.106}$$

and

$$e^{-b\Lambda} = z^2(s_1t_2 - s_2t_1) + z(s_1v_2 - s_2v_1 + u_1t_2 - u_2t_1) + u_1v_2 - u_2v_1 \tag{3.107}$$

Let us show that the relation (3.106) results from the general formula (3.50). Note that one can write the solution (3.104) in the general form (3.23)

$$V_1(z,\bar{z};R_1) = \sqrt{-\lambda}(s_1z\bar{z} + t_1z + u_1\bar{z} + v_1) = \sqrt{-\lambda}[z(s_1\bar{z} + t_1) + (u_1\bar{z} + v_1)] \tag{3.108}$$

with

$$\begin{aligned} a_1 &= z\,, \qquad & a_2 &= 1 \\ b_1 &= s_1\bar{z} + t_1\,, \qquad & b_2 &= u_1\bar{z} + v_1 \end{aligned} \tag{3.109}$$

Remember that topological defects can be obtained rotating the pair a_1, a_2 by a $SL(2,C)$ matrix $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, namely taking

$$\begin{aligned} \tilde{a}_1 &= \alpha z + \beta \\ \tilde{a}_2 &= \gamma z + \delta \end{aligned} \tag{3.110}$$

and keeping the same b_1 and b_2 as in (3.109). Using (3.110) we obtain new solution with $R_2 = DR_1$, Recalling that according to (3.50) $2\kappa = \mathrm{Tr}\ D$ we arrive to (3.106).

We would like to mention also folded version of the defect solution, obeying the permutation brane boundary conditions. One can see that the expressions (3.104) and (3.105) satisfy the permutation branes boundary conditions (3.55)-(3.57) with

$$2\kappa = \mathrm{Tr}(R_2^T R_1^{-1}) = s_1v_2 + s_2v_1 - t_1t_2 - u_1u_2 \tag{3.111}$$

and

$$e^{-b\Lambda} = \tau^2(s_2t_1 - s_1u_2) + \tau(s_2v_1 - s_1v_2 + t_1t_2 - u_1u_2) + t_2v_1 - u_1v_2 \tag{3.112}$$

Note that equations (3.111) and (3.112) are in fact folded version of the corresponding defect expressions (3.106) and (3.107) and derived exchanging $u_2 \leftrightarrow t_2$, as result of the $z_2 \leftrightarrow \bar{z}_2$ exchange. The relation (3.111) can be justified again using general formalism developed in section 2.3.

It is interesting to note that in the parameterization (3.102) for hermitian matrices R_1 and R_2

$$R_1 = \begin{pmatrix} X_0 - X_1 & X_2 + iX_3 \\ X_2 - iX_3 & X_0 + X_1 \end{pmatrix} \quad R_2 = \begin{pmatrix} Y_0 - Y_1 & Y_2 + iY_3 \\ Y_2 - iY_3 & Y_0 + Y_1 \end{pmatrix} \tag{3.113}$$

the defect parameter κ is equal to the Minkowski inner product of the vectors X^μ and Y^μ

$$\kappa = X_0 Y_0 - X_1 Y_1 - X_2 Y_2 - X_3 Y_3 \tag{3.114}$$

We are in position to write two-point correlation function in the presence of a defect:

$$\begin{aligned} &\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2) \rangle^{\text{light}} = \qquad (3.115) \\ &\int_{H_3^+ \times H_3^+} dR_1 dR_2 \delta\Big(\mathrm{Tr}\left(R_2 R_1^{-1}\right) - 2\kappa\Big) V_1^{-2\eta}(z_1, \bar{z}_1; R_1) V_2^{-2\eta}(z_2, \bar{z}_2; R_2) \end{aligned}$$

Here dR_i, $i = 1, 2$ denotes integration measure on the 3D hyperboloid H_3^+. This expression allows to establish conformal invariance of defect two-point function. Let us perform the transformation

$$R_1 \to L R_1 L^\dagger \qquad \text{and} \qquad R_2 \to L R_2 L^\dagger \ , \tag{3.116}$$

where L is a $SL(2, C)$ matrix: $L = \begin{pmatrix} m & n \\ k & l \end{pmatrix}$. Recall the transformation rule of the functions $V^{-2\eta}(z, \bar{z}; R)$ under L:

$$V^{-2\eta}(z, \bar{z}; L R L^\dagger) = \frac{1}{|nz + l|^{4\eta}} V^{-2\eta}\left(\frac{mz + k}{nz + l}, c.c; R\right) \tag{3.117}$$

Performing the change of the integration variables (3.116), using that the δ-function arguments is invariant under (3.116) and the transformation rule (3.117) we obtain

$$\langle V_\alpha(z_1,\bar{z}_1) X V_\alpha(z_2,\bar{z}_2)\rangle^{\text{light}} = \tag{3.118}$$
$$\frac{1}{|nz_1+l|^{4\eta}}\frac{1}{|nz_2+l|^{4\eta}}\Big\langle V_\alpha\left(\frac{mz_1+k}{nz_1+l},c.c.\right) X V_\alpha\left(\frac{mz_2+k}{nz_2+l},c.c.\right)\Big\rangle^{\text{light}}$$

which is the standard consequence of the conformal invariance, when we remember that in the light asymptotic limit $\lim_{b\to 0}\Delta_{\eta b} = \eta$. This calculation shows that the fact that the defect parameter κ is invariant under (3.116) is related to the conformal invariance of the defect two-point function.

Using conformal invariance we can set z_1 to ∞ and z_2 to 0 to derive:

$$\langle V_{b\eta}(z_1,\bar{z}_1) X V_{b\eta}(z_2,\bar{z}_2)\rangle^{\text{light}} = \frac{\lambda^{-2\eta}}{(z_1-z_2)^{2\eta}(\bar{z}_1-\bar{z}_2)^{2\eta}}\times \tag{3.119}$$
$$\int_{H_3^+\times H_3^+} dR_1 dR_2 \delta\Big(\mathrm{Tr}\left(R_2R_1^{-1}\right)-2\kappa\Big)(R_1)_{11}^{-2\eta}(R_2)_{22}^{-2\eta}$$

To calculate this integral we express Hermitian matrices R_1 and R_2 as products

$$R_1 = gg^\dagger, \quad R_2 = \tilde{g}\tilde{g}^\dagger, \quad g,\tilde{g}\in SL(2,C) \tag{3.120}$$

implying that

$$V_1 = \sqrt{-\pi\mu b^2}\Big(|g_{11}z+g_{21}|^2 + |g_{12}z+g_{22}|^2\Big) \tag{3.121}$$

$$V_2 = \sqrt{-\pi\mu b^2}\Big(|\tilde{g}_{11}z+\tilde{g}_{21}|^2 + |\tilde{g}_{12}z+\tilde{g}_{22}|^2\Big) \tag{3.122}$$

At the next step we will parametrize $\tilde{g}$ as a product of matrices g and U:

$$\tilde{g} = gU, \tag{3.123}$$

where U is $SL(2,C)$ matrix

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \qquad \text{and} \qquad u_{11}u_{22} - u_{12}u_{21} = 1 \tag{3.124}$$

Inserting (3.120) and (3.123) in (3.106) we obtain

$$2\kappa = \mathrm{Tr}\, UU^{\dagger} \tag{3.125}$$

This can be understood noting that solutions (3.121) and (3.122) correspond to

$$a_i(z) = g_{1i}z + g_{2i} \qquad \tilde{a}_i(z) = \tilde{g}_{1i}z + \tilde{g}_{2i} \qquad i = 1,2 \tag{3.126}$$

$$b_i(\bar{z}) = \bar{g}_{1i}\bar{z} + \bar{g}_{2i} \qquad \tilde{b}_i(\bar{z}) = \bar{\tilde{g}}_{1i}\bar{z} + \bar{\tilde{g}}_{2i} \qquad i = 1,2 \tag{3.127}$$

It is obvious that

$$\tilde{a}_i = \sum_{j=1}^{2} u_{ji}a_j \tag{3.128}$$

$$\tilde{b}_i = \sum_{j=1}^{2} \bar{u}_{ji}b_j \tag{3.129}$$

We see that passing from g to $\tilde{g} = gU$ brings to the simultaneous rotations of a_i and b_i, $i = 1,2$, by matrices U and $\bar{U}$. Therefore the defect parameter κ is equal indeed to the trace of the product $UU^{\dagger}$. In this variables the integral (3.119) simplifies and reads

$$\begin{aligned} &\langle V_{b\eta}(z_1,\bar{z}_1) X V_{b\eta}(z_2,\bar{z}_2)\rangle^{\text{light}} = \frac{\lambda^{-2\eta}}{(z_1-z_2)^{2\eta}(\bar{z}_1-\bar{z}_2)^{2\eta}} \times \\ &\int dR_1 dU \delta(|u_{11}|^2 + |u_{12}|^2 + |u_{21}|^2 + |u_{22}|^2 - 2\kappa)(R_1)_{11}^{-2\eta}(R_2)_{22}^{-2\eta} \end{aligned} \tag{3.130}$$

where dR_1 and dU corresponding integration measures which will be elaborated below.

Using $SU(2)$ freedom in the choice of g we can adopt the parameterization

$$g = \begin{pmatrix} \rho_1^{-1} & a_1 \\ 0 & \rho_1 \end{pmatrix} \tag{3.131}$$

and

$$R_1 = \begin{pmatrix} \rho_1^{-2} + |a_1|^2 & \rho_1 a_1 \\ \rho_1 \bar{a}_1 & \rho_1^2 \end{pmatrix} \tag{3.132}$$

Parameterizing $\tilde{g}$ in the same way

$$\tilde{g} = \begin{pmatrix} \rho_2^{-1} & a_2 \\ 0 & \rho_2 \end{pmatrix} \tag{3.133}$$

we find that the elements of the matrix $U = g^{-1}\tilde{g}$ satisfy the relations

$$\begin{aligned} & u_{21} = 0 && (3.134) \\ & u_{22} = u_{11}^{-1} \equiv u \qquad u \in \mathbb{R} \\ & \rho_2 = \rho_1 u \\ & a_2 = \rho_1^{-1} u_{12} + a_1 u \end{aligned}$$

$$\tag{3.135}$$

Eq. (3.134) implies

$$R_2 = \begin{pmatrix} \rho_1^{-2}u^{-2} + |\rho_1^{-1}u_{12} + a_1 u|^2 & \rho_1 u(\rho_1^{-1}u_{12} + a_1 u) \\ \rho_1 u(\rho_1^{-1}\bar{u}_{12} + \bar{a}_1 u) & \rho_1^2 u^2 \end{pmatrix} \tag{3.136}$$

Using the volume form on the 3D hyperboloid H_3^+ computed in appendix B (**??**), one obtains for the integration measure

$$dR_1 dR_2 = \rho_1 d\rho_1 d^2 a u du d^2 u_{12} \tag{3.137}$$

Now the integral (3.130) takes the form

$$\langle V_{b\eta}(z_1,\bar z_1) X V_{b\eta}(z_2,\bar z_2)\rangle^{\text{light}} = \frac{\lambda^{-2\eta}}{(z_1-z_2)^{2\eta}(\bar z_1-\bar z_2)^{2\eta}} \times \\ \int \rho_1 d\rho_1 d^2 a u du d^2 u_{12} \delta\left(u^2+\frac{1}{u^2}+|u_{12}|^2-2\kappa\right) \frac{1}{(\rho_1^{-2}+|a_1|^2)^{2\eta}} \frac{1}{\rho_1^{4\eta} u^{4\eta}} \tag{3.138}$$

Performing the integral over u_{12} and then over u we obtain

$$\langle V_{b\eta}(z_1,\bar z_1) X V_{b\eta}(z_2,\bar z_2)\rangle^{\text{light}} = \\ \pi\lambda^{-2\eta} \frac{\left((\kappa+\sqrt{\kappa^2-1})^{1-2\eta}-(\kappa-\sqrt{\kappa^2-1})^{1-2\eta}\right)}{2(1-2\eta)(z_1-z_2)^{2\eta}(\bar z_1-\bar z_2)^{2\eta}} \times \\ \int \rho_1 d\rho_1 d^2 a \frac{1}{(\rho_1^{-2}+|a_1|^2)^{2\eta}} \frac{1}{\rho_1^{4\eta}} \tag{3.139}$$

Performing the integral over a one gets

$$\int \rho_1 d\rho_1 d^2 a \frac{1}{(\rho_1^{-2}+|a_1|^2)^{2\eta}} \frac{1}{\rho_1^{4\eta}} = \frac{1}{2\eta-1}\int \frac{d\rho}{\rho} = \frac{1}{2\eta-1}\delta(0) \tag{3.140}$$

This integral diverges. This divergence was analyzed in [94] and related to the infinite volume of the dilation group. It brings in fact to the $\delta(0)$ which appears in the two-point function of coincident fields of the continuous spectrum. We can get finite result taking the relation

$$\frac{\langle V_{b\eta}(z_1,\bar z_1) X V_{b\eta}(z_2,\bar z_2)\rangle^{\text{light}}}{\langle V_0(z_1,\bar z_1) X V_0(z_2,\bar z_2)\rangle^{\text{light}}} = \frac{\lambda^{-2\eta}\sinh 2\pi\sigma(1-2\eta)}{(1-2\eta)^2(z_1-z_2)^{2\eta}(\bar z_1-\bar z_2)^{2\eta}\sinh 2\pi\sigma} \tag{3.141}$$

Here we set $\kappa=\cosh 2\pi\sigma$.

Using the properties of the Γ functions collected in appendix A one can calculate the light asymptotic limit of the ZZ function (3.72):

$$\frac{W^{-1}_{\alpha=\eta b}}{W^{-1}_{\alpha=0}} \to (\pi\mu b^2)^{-\eta}\frac{1}{1-2\eta} \tag{3.142}$$

and setting $s=\frac{\sigma}{b}$ and $\alpha=\eta b$ we obtain

$$\frac{\cosh 2\pi s(2\alpha - Q)}{\cosh 2\pi s Q} \to e^{-4\pi\eta|\sigma|} \tag{3.143}$$

Hence, recalling (3.80) we get in the light asymptotic limit for the defect two-point function derived via the bootstrap program

$$\frac{\langle V_{b\eta}(z_1,\bar{z}_1) X V_{b\eta}(z_2,\bar{z}_2)\rangle}{\langle V_0(z_1,\bar{z}_1) X V_0(z_2,\bar{z}_2)\rangle} \to \frac{\lambda^{-2\eta}}{(2\eta-1)^2} \frac{e^{-4\pi\eta|\sigma|}}{(z_1-z_2)^{2\eta}(\bar{z}_1-\bar{z}_2)^{2\eta}} \tag{3.144}$$

In the limit of the large σ we get full agreement between (3.141) and (3.144).

3.5 Defects in heavy asymptotic limit

3.5.1 Heavy asymptotic limit of the correlation functions

In this section we consider the heavy asymptotic limit of the two-point functions in the presence of defects (3.80). Now we should compute the inverse ZZ function (3.72) and the factor $\cosh(2\pi s(2\alpha - Q))$ in the limit $b \to 0$, setting $\alpha = \frac{\eta}{b}$, and $s = \frac{\sigma}{b}$. In the heavy asymptotic limit we should keep only terms having the form $\sim e^{1/b^2}$.

Here we find very useful to consider in the spirit of [99] analytic continuation of the Liouville theory with complex η and complex saddle points.

Taking the η satisfying the Seiberg bound (3.90), using properties of Γ functions collected in appendix A, and keeping only terms important in the heavy asymptotic limit we obtain

$$W^{-1}_{\alpha=\frac{\eta}{b}} \sim \lambda^{\frac{1-2\eta}{2b^2}} \frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} \exp\left(\frac{2\eta-1}{b^2}\Big[\ln(1-2\eta)-1\Big]\right) \tag{3.145}$$

The importance of the term $\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)}$ is explained in [99]. It was shown there that this term in the semiclassical interpretation arises as sum over some "instanton" like sectors. As a preparation to this point we will expand this term in two ways as suggested in [99]. Denoting

$y = e^{i\pi(2\eta-1)/b^2}$ one can write

$$\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} = \frac{2i}{y-y^{-1}} = -2i\sum_{k=0}^{\infty} y^{-(2k+1)} = 2i\sum_{k=0}^{\infty} y^{2k+1} \tag{3.146}$$

One expansion is valid for $|y|>1$ and one for $|y|<1$. So either way, there is a set T of integers with

$$\frac{1}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} = 2i\sum_{M\in T} e^{2i\pi(M\mp 1/2)(2\eta-1)/b^2} \tag{3.147}$$

T consists of nonnegative integers if $\mathrm{Im}(2\eta-1)/b^2>0$ and of nonpositive ones if $\mathrm{Im}(2\eta-1)/b^2<0$.

Setting $\alpha=\frac{\eta}{b}$ and $s=\frac{\sigma}{b}$ we easily obtain:

$$\cosh 2\pi s(2\alpha-Q)\to e^{\frac{2}{b^2}\pi|\sigma|(1-2\eta)} \tag{3.148}$$

Now we are position to write down the limiting form of the defects correlation functions. Inserting (3.145), (3.148) in (3.80) we can write in the heavy asymptotic limit

$$\begin{aligned}\langle V_\alpha(z_1,\bar z_1)XV_\alpha(z_2,\bar z_2)\rangle &\sim (z_1-z_2)^{-2\eta(1-\eta)/b^2}(\bar z_1-\bar z_2)^{-2\eta(1-\eta)/b^2}\times \\ \lambda^{\frac{1-2\eta}{b^2}}&\frac{1}{\sin^2\pi\left(\frac{2\eta-1}{b^2}\right)}\exp\left(\frac{4\eta-2}{b^2}\Big[\ln(1-2\eta)-1\Big]\right)e^{\frac{2}{b^2}\pi|\sigma|(1-2\eta)}\end{aligned} \tag{3.149}$$

Using also (3.147) we get

$$\langle V_\alpha(z_1,\bar z_1)XV_\alpha(z_2,\bar z_2)\rangle \sim \sum_{M_1,M_2\in T}\exp\left(-S^{\mathrm{def}}_{M_1,M_2}\right) \tag{3.150}$$

where

$$\begin{aligned}b^2 S^{\mathrm{def}}_{M_1,M_2} &= -2i\pi(M_1+M_2\mp 1)(2\eta-1)+4\eta(1-\eta)\log|z_1-z_2|- \\ &(1-2\eta)\log\lambda-(4\eta-2)\log(1-2\eta)+(4\eta-2)-2\pi|\sigma|(1-2\eta)\end{aligned} \tag{3.151}$$

It is instructive to compare the heavy asymptotic limit of the defect two-point function with the corresponding limit of usual two-point function, computed in [99]

$$\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\rangle \sim |z_1-z_2|^{-4\eta(1-\eta)/b^2}\times \\ \lambda^{(1-2\eta)/b^2}\frac{1}{\sin\pi(2\eta-1)/b^2}\exp\left(\frac{4\eta-2}{b^2}[\ln(1-2\eta)-1]\right) \tag{3.152}$$

The relation of (3.149) to (3.152) naturally gives the heavy asymptotic limit of the eigenvalues $D(\alpha)$ of the defect operator:

$$D(\alpha)=\frac{\langle V_\alpha(z_1,\bar{z}_1)XV_\alpha(z_2,\bar{z}_2)\rangle}{\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\rangle}\to\frac{e^{\frac{2}{b^2}\pi|\sigma|(1-2\eta)}}{\sin\pi\left(\frac{2\eta-1}{b^2}\right)} \tag{3.153}$$

3.5.2 Evaluation of the action for classical solutions

According to general prescription of the semiclassical heavy asymptotic limit, we should find solutions of the Liouville equation, satisfying the defect equations of motion and possessing the logarithmic singularities (3.89) at points z_1 and z_2. The form of the solution of the defect equations of motions (3.41) and (3.42) implies that we should find functions $A(z)$, $C(z)$ and $B(\bar{z})$ in such a way that ϕ_1 has logarithmic singularity at point z_1 and and ϕ_2 has logarithmic singularity at point z_2. Since the energy-momentum tensor is continuous across a defect this implies that we should find solutions possessing two singular points. Two-point solutions are well known (see for example [99]) and we can build from them the ansatz satisfying the defect equations of motion.

Let us take as $A(z)$

$$A(z)=e^{2t_1}(z-z_1)^{2\eta-1}(z-z_2)^{1-2\eta} \tag{3.154}$$

One has also

$$a_1=\frac{1}{\sqrt{\partial A}}=\frac{e^{-t_1}}{\sqrt{(z_1-z_2)(2\eta-1)}}(z-z_1)^{1-\eta}(z-z_2)^{\eta} \tag{3.155}$$

$$a_2=\frac{A}{\sqrt{\partial A}}=\frac{e^{t_1}}{\sqrt{(z_1-z_2)(2\eta-1)}}(z-z_1)^{\eta}(z-z_2)^{1-\eta} \tag{3.156}$$

Inserting (3.155) or (3.156) in (3.17) we obtain the energy-momentum tensor

$$b^2T = \frac{\eta(1-\eta)}{(z-z_1)^2} + \frac{\eta(1-\eta)}{(z-z_2)^2} + \frac{2\eta(1-\eta)}{z_1-z_2}\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) \tag{3.157}$$

indeed possessing two singular points.

The anti-holomorphic part is essentially the same with η replaced by $1-\eta$:

$$B(\bar{z}) = (\bar{z}-\bar{z}_1)^{1-2\eta}(\bar{z}-\bar{z}_2)^{2\eta-1} \tag{3.158}$$

$$b_1 = \frac{B}{\sqrt{\partial B}} = \frac{1}{\sqrt{(\bar{z}_1-\bar{z}_2)(1-2\eta)}}(\bar{z}-\bar{z}_1)^{1-\eta}(\bar{z}-\bar{z}_2)^{\eta} \tag{3.159}$$

$$b_2 = -\frac{1}{\sqrt{\partial B}} = -\frac{1}{\sqrt{(\bar{z}_1-\bar{z}_2)(1-2\eta)}}(\bar{z}-\bar{z}_1)^{\eta}(\bar{z}-\bar{z}_2)^{1-\eta} \tag{3.160}$$

Let us take the holomorphic part for ϕ_2 as

$$C(z) = e^{2t_2}(z-z_1)^{2\eta-1}(z-z_2)^{1-2\eta} = e^{2(t_2-t_1)}A(z) \tag{3.161}$$

and the antiholomorphic part again given by (3.158). Using (3.45) one gets

$$\kappa = \cosh(t_2-t_1) \tag{3.162}$$

Inserting (3.154), (3.161) and (3.158) in (3.41) and (3.42) we obtain:

$$e^{-\varphi_1} = -\frac{\lambda}{(2\eta-1)^2|z_1-z_2|^2}\left(e^{t_1}|z-z_1|^{2\eta}|z-z_2|^{2-2\eta} + e^{-t_1}|z-z_1|^{2-2\eta}|z-z_2|^{2\eta}\right)^2 \tag{3.163}$$

$$e^{-\varphi_2} = -\frac{\lambda}{(2\eta-1)^2|z_1-z_2|^2}\left(e^{t_2}|z-z_1|^{2\eta}|z-z_2|^{2-2\eta} + e^{-t_2}|z-z_1|^{2-2\eta}|z-z_2|^{2\eta}\right)^2 \tag{3.164}$$

It is easy to see that φ_1 and φ_2 given by (3.163) and (3.164) have the required singularity around z_1 and z_2 respectively. In fact each of the functions φ_1 or φ_2 given by (3.163) and (3.164) coincides with the solution describing saddle point for two-point function considered in [99]. But in [99] this solution was considered on a full plane with the same parameter t

everywhere, whereas here each of them is considered on a corresponding half-plane, namely in (3.163) z belongs to the upper half-plane Σ_1, and in (3.164) z belongs to the lower half-plane Σ_2, and we should also remember that, $z_1 \in \Sigma_1$ and $z_2 \in \Sigma_2$. The defect is created by the choice of different parameters t_1 and t_2, $t_1 \neq t_2$.

To evaluate the action on solutions (3.163), (3.164), we will use the strategy used in [95]. Namely we will write the system of differential equations which this action should satisfy. The first equation is (3.95) which given that $\eta_1 = \eta_2 = \eta$ reads

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \eta} = -X_1 - X_2 \tag{3.165}$$

where X_i defined in (3.89). The leading terms of φ_1 around z_1 are

$$\varphi_1 \to -4\eta \log |z - z_1| + X_1 \tag{3.166}$$

where

$$X_1 = 2\pi i \left(2N_1 + \frac{1}{2}\right) - \log \lambda + 2\log(1-2\eta) - (2-4\eta)\log|z_1 - z_2| - 2t_1 \tag{3.167}$$

Here N_1 is an integer. The possibility to add the term $4i\pi N_1$ results from the invariance of the action (3.98) under the transformation $\varphi_i \to \varphi + 4\pi i N_i$, $i = 1, 2$. Note that the corresponding transformation in the bulk Liouville theory reads $\varphi \to \varphi + 2\pi i N$, and broken to the $\varphi \to \varphi + 4\pi i N$ due to presence of exponential terms on the defect.

The leading terms of φ_2 around z_2 similarly are

$$\varphi_2 \to -4\eta \log |z - z_2| + X_2 \tag{3.168}$$

where

$$X_2 = 2\pi i \left(2N_2 + \frac{3}{2}\right) - \log \lambda + 2\log(1-2\eta) - (2-4\eta)\log|z_1 - z_2| + 2t_2 \tag{3.169}$$

where $N_2 \in \mathbb{Z}$. Inserting (3.167) and (3.169) in (3.165) one obtains

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \eta} = -2\pi i \left(2N_1 + 2N_2 + 2\right) + 2 \log \lambda - 4 \log(1-2\eta) + (4-8\eta) \log |z_1 - z_2| + 2(t_1 - t_2) \quad (3.170)$$

Here we would like to emphasize yet another difference from the two-point function calculation in [99]. In two-point calculation the integers N_1 and N_2 are equal since we have one continuous function ϕ. Here they can be different since we have two different functions φ_1 and φ_2.

The action with defect (3.98) implies also

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \kappa} = \frac{1}{i\pi} \int_{\partial \Sigma_1} e^{\Lambda b} \quad (3.171)$$

Inserting (3.154) and (3.161) in eq. (3.44) one obtains

$$e^{\Lambda b} = \frac{1}{2 \sinh(t_1 - t_2)} \frac{(2\eta - 1)(z_1 - z_2)}{(z - z_1)(z - z_2)} \quad (3.172)$$

Using that

$$\frac{1}{i} \int_{\partial \Sigma_1} \frac{dz}{(z - z_1)(z - z_2)} = \frac{2\pi}{(z_1 - z_2)} \quad (3.173)$$

we obtain

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \kappa} = \frac{2\eta - 1}{\sinh(t_1 - t_2)} \quad (3.174)$$

The Polyakov's relation (3.96) additionally implies

$$b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial z_i} = \pm \frac{2\eta(1 - \eta)}{z_1 - z_2} \qquad i = 1, 2 \quad (3.175)$$

Integrating equations (3.170), (3.174) and (3.175) we obtain:

$$b^2 S_{N_1,N_2}^{\text{def}} = -2i\pi(2N_1 + 2N_2 + 2)\eta + 4\eta(1 - \eta) \log |z_1 - z_2| + \quad (3.176)$$

$$2\eta \log \lambda - (4\eta - 2) \log(1 - 2\eta) + 4\eta - (t_1 - t_2)(1 - 2\eta) + C$$

where C is a constant. To derive the penultimate term we should remember the relation (3.162). To fix the constant term we can directly compute the action (3.98) for the ansatz (3.163)-(3.164) with $\eta = 0$

$$\varphi_1 = i\pi + 4i\pi N_1 - \log\lambda - 2\log\left(\frac{e^{t_1}}{|z_1 - z_2|}|z - z_2|^2 + \frac{e^{-t_1}}{|z_1 - z_2|}|z - z_1|^2\right) \tag{3.177}$$

$$\varphi_2 = 3i\pi + 4i\pi N_2 - \log\lambda - 2\log\left(\frac{e^{t_2}}{|z_1 - z_2|}|z - z_2|^2 + \frac{e^{-t_2}}{|z_1 - z_2|}|z - z_1|^2\right) \tag{3.178}$$

The solutions (3.177) and (3.178) can be derived by the $SL(2, C)$ conformal transformation $z \to \frac{z - z_1}{z - z_2}$ from the solutions:

$$\varphi_1 = i\pi + 4i\pi N_1 - \log\lambda - 2\log\left(e^{t_1} + e^{-t_1} z\bar{z}\right) \tag{3.179}$$

$$\varphi_2 = 3i\pi + 4i\pi N_2 - \log\lambda - 2\log\left(e^{t_2} + e^{-t_2} z\bar{z}\right) \tag{3.180}$$

Evaluating the action (3.98) on the ansatz (3.179), (3.180) we obtain

$$b^2 S_0 = 2i\pi(N_1 + N_2 + 1) - \log\lambda - 2 - (t_1 - t_2) \tag{3.181}$$

Comparing (3.181) with (3.176) fixes the constant C:

$$C = 2i\pi(N_1 + N_2 + 1) - \log\lambda - 2 \tag{3.182}$$

Inserting this value of C in (3.176) we indeed obtain (3.151) if we set

$$N_i = M_i, \qquad i = 1, 2 \tag{3.183}$$

and

$$2\pi\sigma = t_1 - t_2 \tag{3.184}$$

The discussion above of the difference points between calculation of two-point function with and without defect suggests nice interpretation of the defect operator. We have seen that there exist two sources of discontinuity giving rise to the corresponding terms in the defect operators. The heavy asymptotic limit of $D(\alpha)$ (3.153) has numerator and denominator. The exponential term in numerator as we have seen originates from the discontinuity created by the choice of the different parameters t_1 and t_2. The denominator $\sin\pi\left(\frac{2\eta-1}{b^2}\right)$ reflects the possibility of the choice of different logarithmic branches. The final quantum expression (3.81) results from the quantum corrections restoring $b \leftrightarrow b^{-1}$ duality of the Liouville theory.

Let us analyze in the heavy asymptotic limit also the relation (3.82) between parameter s and $A(b)$

$$2\cosh 2\pi bs = A(b)\left(\frac{W(-b/2)}{W(0)}\right)^2. \tag{3.185}$$

It is easy to compute that

$$\mathrm{lim}_{\mathrm{b}\to 0}\frac{\mathrm{W}(-\mathrm{b}/2)}{\mathrm{W}(0)} = -\frac{2}{\sqrt{\lambda}} \tag{3.186}$$

Setting that $s = \frac{\sigma}{b}$, we get

$$\cosh 2\pi\sigma = \frac{2A(0)}{\lambda} \tag{3.187}$$

This implies that parameter κ is proportional to $A(0)$:

$$\kappa = \frac{2A(0)}{\lambda} \tag{3.188}$$

Note that as in the light asymptotic limit as well as in the heavy asymptotic limit we get the same relation between σ and κ

$$\kappa = \cosh 2\pi\sigma \tag{3.189}$$

Chapter 4

Comments on fusion matrix in N=1 super Liouville fieldtheory

4.1 N=1 Super Liouville field theory

Let us review basic facts on the $N = 1$ Super Liouville field theory. Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{2\pi} g_{ab}\partial_a\varphi\partial_b\varphi + \frac{1}{2\pi}(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) + 2i\mu b^2\bar{\psi}\psi e^{b\varphi} + 2\pi\mu^2 b^2 e^{2b\varphi}\,, \tag{4.1}$$

The energy-momentum tensor and the superconformal current are

$$T = -\frac{1}{2}(\partial\varphi\partial\varphi - Q\partial^2\varphi + \psi\partial\psi) \tag{4.2}$$

$$G = i(\psi\partial\varphi - Q\partial\psi) \tag{4.3}$$

The superconformal algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \tag{4.4}$$

$$[L_m, G_k] = \frac{m-2k}{2}G_{m+k} \tag{4.5}$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l} \tag{4.6}$$

with the central charge

$$c_L = \frac{3}{2} + 3Q^2 \,. \tag{4.7}$$

where

$$Q = b + \frac{1}{b} \,. \tag{4.8}$$

where k and l take integer values for the Ramond algebra and half-integer values for the Neveu-Schwarz algebra.

NS-NS primary fields $N_\alpha(z, \bar{z})$ in this theory, $N_\alpha(z, \bar{z}) = e^{\alpha\varphi(z,\bar{z})}$, have conformal dimensions

$$\Delta_\alpha^{NS} = \frac{1}{2}\alpha(Q - \alpha) \,. \tag{4.9}$$

The physical states have $\alpha = \frac{Q}{2} + iP$.

Introduce also the field

$$\tilde{N}_\alpha(z, \bar{z}) = G_{-1/2}\bar{G}_{-1/2}N_\alpha(z, \bar{z}) \tag{4.10}$$

The R-R is defined as

$$R_\alpha(z, \bar{z}) = \sigma(z, \bar{z})e^{\alpha\varphi(z,\bar{z})} \tag{4.11}$$

where σ is the spin field.

The dimension of the R-R operator is

$$\Delta_\alpha^R = \frac{1}{16} + \frac{1}{2}\alpha(Q - \alpha) \tag{4.12}$$

The NS-NS and R-R operators with the same conformal dimensions are proportional to each other, namely we have

$$N_\alpha = \mathcal{G}_{NS}(\alpha)N_{Q-\alpha} \tag{4.13}$$

$$R_\alpha = \mathcal{G}_R(\alpha)R_{Q-\alpha} \tag{4.14}$$

$\mathcal{G}_{NS}(\alpha)$ and $\mathcal{G}_R(\alpha)$ are called reflection functions. They also give two-point functions. The elegant way to write the reflection functions is to introduce NS and R generalization of the ZZ function:

$$W_{NS}(\alpha) = \frac{2(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}\pi(\alpha - Q/2)}{\Gamma(1 + b(\alpha - Q/2))\Gamma(1 + \frac{1}{b}(\alpha - Q/2))} \tag{4.15}$$

$$W_R(\alpha) = \frac{2\pi(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}}{\Gamma(1/2 + b(\alpha - Q/2))\Gamma(1/2 + \frac{1}{b}(\alpha - Q/2))} \tag{4.16}$$

The reflection functions can be written

$$\mathcal{G}_{NS}(\alpha) = \frac{W^{NS}(Q-\alpha)}{W^{NS}(\alpha)} \tag{4.17}$$

$$\mathcal{G}_R(\alpha) = \frac{W^R(Q-\alpha)}{W^R(\alpha)} \tag{4.18}$$

The functions (4.15) and (4.16) satisfy also the relations

$$W_{NS}(\alpha)W_{NS}(Q-\alpha) = -4\sin\pi b(\alpha - Q/2)\sin\pi\frac{1}{b}(\alpha - Q/2) \tag{4.19}$$

$$W_R(\alpha)W_R(Q-\alpha) = 4\cos\pi b(\alpha - Q/2)\cos\pi\frac{1}{b}(\alpha - Q/2) \tag{4.20}$$

The degenerate states are given by the momenta:

$$\alpha_{m,n} = \frac{1}{2b}(1-m) + \frac{b}{2}(1-n) \tag{4.21}$$

with even $m-n$ in the NS sector and odd $m-n$ in the R sector.

For the super conformal theory, characters are defined for the NS sector, for the R sector and the $\widetilde{\mathrm{NS}}$ sector. The corresponding characters for generic P which have no null-states are

$$\chi_P^{NS}(\tau) = \sqrt{\frac{\theta_3(q)}{\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{4.22}$$

$$\chi_P^{\widetilde{NS}}(\tau) = \sqrt{\frac{\theta_4(q)}{\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{4.23}$$

$$\chi^R_P(\tau) = \sqrt{\frac{\theta_2(q)}{2\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{4.24}$$

where $q = \exp(2\pi i\tau)$ and

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{4.25}$$

Modular transformation of characters (4.22) - (4.24) is well-known:

$$\chi^{NS}_P(\tau) = \int \chi^{NS}_{P'}(-1/\tau)e^{-2i\pi PP'}dP'\,. \tag{4.26}$$

$$\chi^{\widetilde{NS}}_P(\tau) = \int \chi^{R}_{P'}(-1/\tau)e^{-2i\pi PP'}dP'\,. \tag{4.27}$$

$$\chi^{R}_P(\tau) = \int \chi^{\widetilde{NS}}_{P'}(-1/\tau)e^{-2i\pi PP'}dP'\,. \tag{4.28}$$

For degenerate representations, the characters are given by those of the corresponding Verma modules subtracted by those of null submodules:

$$\chi^{NS}_{m,n} = \chi^{NS}_{\frac{1}{2}(nb+mb^{-1})} - \chi^{NS}_{\frac{1}{2}(nb-mb^{-1})} \tag{4.29}$$

$$\chi^{\widetilde{NS}}_{m,n} = \chi^{\widetilde{NS}}_{\frac{1}{2}(nb+mb^{-1})} - (-)^{rs}\chi^{\widetilde{NS}}_{\frac{1}{2}(nb-mb^{-1})} \tag{4.30}$$

$$\chi^{R}_{m,n} = \chi^{R}_{\frac{1}{2}(nb+mb^{-1})} - \chi^{R}_{\frac{1}{2}(nb-mb^{-1})} \tag{4.31}$$

Modular transformations of (4.29) - 4.31) are

$$\chi^{NS}_{m,n}(\tau) = \int \chi^{NS}_{P}(-1/\tau)2\sinh(\pi mP/b)\sinh(\pi nbP)dP\,. \tag{4.32}$$

$$\chi^{\widetilde{NS}}_{m,n}(\tau) = \int \chi^{R}_{P}(-1/\tau)2\sinh(\pi mP/b)\sinh(\pi nbP)dP\,, \quad m,n \quad \text{even} \tag{4.33}$$

$$\chi^{\widetilde{NS}}_{m,n}(\tau) = \int \chi^{R}_{P}(-1/\tau)2\cosh(\pi mP/b)\cosh(\pi nbP)dP\,. \quad m,n \quad \text{odd} \tag{4.34}$$

Note that the vacuum component of the matrix of modular transformation specified by $(m,n) = (1,1)$ in formulae (4.32) - (4.34) coincide with the right hand side of (4.19) and (4.20) similar to the bosonic Liouville theory.

The structure constants in $N = 1$ super Liouville field theory are computed in [69, 70]:

$$\langle N_{\alpha_1}(z_1, \bar{z}_1) N_{\alpha_2}(z_2, \bar{z}_2) N_{\alpha_3}(z_3, \bar{z}_3)\rangle = \\ \frac{C_{NS}(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3})}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1})}|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2})}} \tag{4.35}$$

$$\langle \tilde{N}_{\alpha_1}(z_1, \bar{z}_1) N_{\alpha_2}(z_2, \bar{z}_2) N_{\alpha_3}(z_3, \bar{z}_3)\rangle = \\ \frac{\tilde{C}_{NS}(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3}+1/2)}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1}-1/2)}|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2}+1/2)}} \tag{4.36}$$

$$\langle R_{\alpha_1}(z_1, \bar{z}_1) R_{\alpha_2}(z_2, \bar{z}_2) N_{\alpha_3}(z_3, \bar{z}_3)\rangle = \\ \frac{C_R(\alpha_1, \alpha_2|\alpha_3) + \tilde{C}_R(\alpha_1, \alpha_2|\alpha_3)}{|z_{12}|^{2(\Delta^R_{\alpha_1}+\Delta^R_{\alpha_2}-\Delta^N_{\alpha_3})}|z_{23}|^{2(\Delta^R_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^R_{\alpha_1})}|z_{13}|^{2(\Delta^R_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^R_{\alpha_2})}} \tag{4.37}$$

where $z_{ij} = z_i - z_j$,

and

$$C_{NS}(\alpha_1, \alpha_2, \alpha_3) = \lambda^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times \\ \frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{NS}(\alpha_1+\alpha_2-\alpha_3)\Upsilon_{NS}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{NS}(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.38}$$

$$\tilde{C}_{NS}(\alpha_1, \alpha_2, \alpha_3) = \lambda^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times \\ \frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_R(\alpha_1+\alpha_2-\alpha_3)\Upsilon_R(\alpha_2+\alpha_3-\alpha_1)\Upsilon_R(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.39}$$

$$C_R(\alpha_1, \alpha_2|\alpha_3) = \lambda^{(Q-\sum_{i=1}^3 \alpha_i)/b} \times \\ \frac{\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_R(\alpha_1+\alpha_2-\alpha_3)\Upsilon_{NS}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{NS}(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.40}$$

$$\tilde{C}_R(\alpha_1,\alpha_2|\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b} \times \tag{4.41}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{NS}(\alpha_1+\alpha_2-\alpha_3)\Upsilon_R(\alpha_2+\alpha_3-\alpha_1)\Upsilon_R(\alpha_3+\alpha_1-\alpha_2)},$$

and

$$\lambda = \pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2} \tag{4.42}$$

Fusion matrix in NS sector is computed in [71, 72]. Let us denote

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^1 \equiv F_{N_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}, \quad F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^2 \equiv F_{N_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix} \tag{4.43}$$

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^1 \equiv F_{\tilde{N}_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}, \quad F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^2 \equiv F_{\tilde{N}_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix} \tag{4.44}$$

To write the fusion matrix we use the following convention. The functions $\Upsilon_i, \Gamma_i, S_i$ will be understood $\Upsilon_{NS}, \Gamma_{NS}, S_{NS}$ for $i = 1$ mod 2, and $\Upsilon_R, \Gamma_R, S_R$ for $i = 0$ mod 2. Now we can write the fusion matrix:

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_j^i = \tag{4.45}$$
$$\frac{\Gamma_i(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_i(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_i(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_i(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_j(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_j(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_j(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_j(\alpha_s+\alpha_1-\alpha_2)}$$
$$\times\quad\frac{\Gamma_i(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_i(\alpha_1+\alpha_4-\alpha_t)\Gamma_i(\alpha_t+\alpha_4-\alpha_1)\Gamma_i(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_j(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_j(\alpha_3+\alpha_4-\alpha_s)\Gamma_j(\alpha_s+\alpha_4-\alpha_3)\Gamma_j(\alpha_s+\alpha_3+\alpha_4-Q)}$$
$$\times\frac{\Gamma_{\mathrm{NS}}(2Q-2\alpha_s)\Gamma_{\mathrm{NS}}(2\alpha_s)}{\Gamma_{\mathrm{NS}}(Q-2\alpha_t)\Gamma_{\mathrm{NS}}(2\alpha_t-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty}d\tau\, J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_j^i$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^1 = \tag{4.46}$$

$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$
$$+\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^1 = \tag{4.47}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$
$$-\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^2 = \tag{4.48}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$
$$-\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^2 = \tag{4.49}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$
$$+\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$

4.2 Values of fusion matrix for intermediate vacuum states

4.2.1 $\alpha_s \to 0$

Motivated by the form of structure constants (4.38)-(4.41) and fusing matrix (4.45) we define the following general expressions for the fusion matrix:

$$F^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \frac{M^{\mathcal{I}}}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} \tag{4.50}$$

with

$$\begin{aligned} M^{\mathcal{I}} = & \qquad\qquad (4.51)\\ & \frac{\Gamma_A(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_B(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_C(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_D(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_E(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_{NS}(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_E(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_{NS}(\alpha_s+\alpha_1-\alpha_2)}\\ \times\ & \frac{\Gamma_B(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_C(\alpha_1+\alpha_4-\alpha_t)\Gamma_D(\alpha_t+\alpha_4-\alpha_1)\Gamma_A(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_{NS}(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_F(\alpha_3+\alpha_4-\alpha_s)\Gamma_{NS}(\alpha_s+\alpha_4-\alpha_3)\Gamma_F(\alpha_s+\alpha_3+\alpha_4-Q)}\\ & \times\frac{\Gamma_{\mathrm{NS}}(2Q-2\alpha_s)\Gamma_{\mathrm{NS}}(2\alpha_s)}{\Gamma_{\mathrm{L}}(Q-2\alpha_t)\Gamma_{\mathrm{L}}(2\alpha_t-Q)} \end{aligned}$$

$$\begin{aligned} & J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \qquad\qquad (4.52)\\ & \frac{S_{\nu_1}(Q+\tau-\alpha_1)S_{\mathrm{K}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2}(\tau+\alpha_1)S_{\nu_3}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1+1}(Q+\tau+\alpha_4-\alpha_t)S_{\mu_2+1}(\tau+\alpha_4+\alpha_t)S_{\mu_3+1}(Q+\tau+\alpha_2-\alpha_s)S_{\mathrm{K}}(\tau+\alpha_2+\alpha_s)}\\ + \ & \eta\frac{S_{\nu_1+1}(Q+\tau-\alpha_1)S_{\mathrm{K}+1}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2+1}(\tau+\alpha_1)S_{\nu_3+1}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1}(Q+\tau+\alpha_4-\alpha_t)S_{\mu_2}(\tau+\alpha_4+\alpha_t)S_{\mu_3}(Q+\tau+\alpha_2-\alpha_s)S_{\mathrm{K}+1}(\tau+\alpha_2+\alpha_s)} \end{aligned}$$

where $\eta = (-1)^{(1+\sum_i(\nu_i+\mu_i))/2}$. $\mathcal{I}$ denotes fusion matrices of different structures, and capital Latin letters here take values NS and R.

Define also the following general expression for structure constants:

$$C_{\mathcal{I}}(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b} \times \tag{4.53}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_L(2\alpha_1)\Upsilon_E(2\alpha_2)\Upsilon_F(2\alpha_3)}{\Upsilon_A(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_B(\alpha_1+\alpha_2-\alpha_3)\Upsilon_C(\alpha_2+\alpha_3-\alpha_1)\Upsilon_D(\alpha_3+\alpha_1-\alpha_2)},$$

Now consider the limit:

$$\alpha_s = \epsilon \to 0, \qquad \alpha_3 = \alpha_4, \qquad \alpha_1 = \alpha_2\,. \tag{4.54}$$

In this limit using formulae from appendix and the definition (4.53) we get for the factor in front of integral:

$$M^{\mathcal{I}} \to C_{\mathcal{I}}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_F(\alpha_3)W_L(\alpha_t)}{2\pi W_E(Q-\alpha_1)} \times \tag{4.55}$$
$$\frac{S_B(Q-\alpha_t+\alpha_3-\alpha_1)S_D(\alpha_3+\alpha_t-\alpha_1)S_{\mathrm{E}}(2\alpha_1)}{S_{\mathrm{F}}(2\alpha_3)S_{NS}(\epsilon)}$$

Let us now evaluate the integral part of (4.50) in the limit (4.54). For this purpose we will use the formula [81]

$$\sum_{\nu=0,1}(-1)^{\nu(1+\sum_i(\nu_i+\mu_i))/2}\int\frac{dx}{i}\prod_{i=1}^{3}S_{\nu+\nu_i}(x+a_i)S_{1+\nu+\mu_i}(-x+b_i) = 2\prod_{i,j=1}S_{\nu_i+\mu_j}(a_i+b_j) \tag{4.56}$$

$$\sum_i(\nu_i+\mu_i) = 1 \bmod 2 \tag{4.57}$$

and

$$\sum_i(a_i+b_i) = Q \tag{4.58}$$

First note that in the limit (4.54) the arguments of S_K's in numerator and denominator coincide and they get canceled.

For the rest of S's in this limit we get for a_i in the argument of $S_{\nu_i}(\tau+a_i)$ and b_i in the argument of $S_{\mu_i+1}(-\tau + b_i)$:

$$\begin{aligned} a_1 &= Q - \alpha_1 & b_1 &= \alpha_t - \alpha_3 \\ a_2 &= \alpha_1 & b_2 &= Q - \alpha_3 - \alpha_t \\ a_3 &= 2\alpha_3 + \alpha_1 - Q & b_3 &= -\alpha_1 \end{aligned} \tag{4.59}$$

From (4.59) we obtain

$$\begin{aligned} a_1 + b_1 &= Q - \alpha_1 + \alpha_t - \alpha_3 \\ a_1 + b_2 &= 2Q - \alpha_1 - \alpha_3 - \alpha_t \\ a_1 + b_3 &= Q - 2\alpha_1 \end{aligned} \tag{4.60}$$

$$\begin{aligned} a_2 + b_1 &= \alpha_1 + \alpha_t - \alpha_3 \\ a_2 + b_2 &= Q + \alpha_1 - \alpha_3 - \alpha_t \\ a_2 + b_3 &= \epsilon \end{aligned} \tag{4.61}$$

$$\begin{aligned} a_3 + b_1 &= \alpha_3 + \alpha_t + \alpha_1 - Q \\ a_3 + b_2 &= \alpha_1 + \alpha_3 - \alpha_t \\ a_3 + b_3 &= 2\alpha_3 - Q \end{aligned} \tag{4.62}$$

Note that

$$\begin{aligned} a_1 + b_1 &= Q - (a_3 + b_2) \\ a_1 + b_2 &= Q - (a_3 + b_1) \end{aligned} \tag{4.63}$$

and

$$\sum_i (a_i + b_i) = Q \tag{4.64}$$

Let us impose also

$$\begin{aligned} &\nu_1 + \mu_1 = \nu_3 + \mu_2 \mod 2 \\ &\nu_1 + \mu_2 = \nu_3 + \mu_1 \mod 2 \\ &\nu_2 + \mu_3 = 1 \mod 2 \end{aligned} \tag{4.65}$$

Assuming also that (4.57) is satisfied we get from (4.56) using formulas (4.60)-(4.65)

$$\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau J^{\mathcal{I}}_{\alpha_s,\alpha_t}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ & \\ \alpha_4 & \alpha_1 \end{array}\right] \to \frac{2S_{\nu_2+\mu_1}(\alpha_1+\alpha_t-\alpha_3)S_{\nu_3+\mu_3}(2\alpha_3-Q)S_{\mathrm{NS}}(\epsilon)}{S_{\nu_1+\mu_3}(2\alpha_1)S_{\nu_2+\mu_2}(\alpha_3+\alpha_t-\alpha_1)} \tag{4.66}$$

Requiring additionally that

$$\begin{aligned} \nu_2 + \mu_1 &= B \\ \nu_2 + \mu_2 &= D \\ \nu_1 + \mu_3 &= E \\ \nu_3 + \mu_3 &= F \end{aligned} \tag{4.67}$$

where these equalities as before understood in a sense, that odd sums identified with the NS sector, and even sums identified with the Ramond sectors, we get

$$F^{\mathcal{I}}_{0,\alpha_t}\left[\begin{array}{cc} \alpha_3 & \alpha_1 \\ & \\ \alpha_3 & \alpha_1 \end{array}\right] = C_{\mathcal{I}}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_L(\alpha_t)}{\pi W_E(Q-\alpha_1)W_F(Q-\alpha_3)} \tag{4.68}$$

4.2.2 $\alpha_t \to 0$ limit

Consider the same fusing matrix, but parametrized in the form

$$F^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \frac{\mathcal{R}^{\mathcal{I}}}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} \tag{4.69}$$

with

$$\begin{aligned}
&\mathcal{R}^{\mathcal{I}} = \\
&\frac{\Gamma_E(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_{NS}(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_E(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_{NS}(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_A(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_B(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_C(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_D(\alpha_s+\alpha_1-\alpha_2)}\\
\times\ &\frac{\Gamma_{NS}(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_F(\alpha_1+\alpha_4-\alpha_t)\Gamma_{NS}(\alpha_t+\alpha_4-\alpha_1)\Gamma_F(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_B(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_C(\alpha_3+\alpha_4-\alpha_s)\Gamma_D(\alpha_s+\alpha_4-\alpha_3)\Gamma_A(\alpha_s+\alpha_3+\alpha_4-Q)}\\
&\times\frac{\Gamma_{\mathrm{L}}(2Q-2\alpha_s)\Gamma_{\mathrm{L}}(2\alpha_s)}{\Gamma_{\mathrm{NS}}(Q-2\alpha_t)\Gamma_{\mathrm{NS}}(2\alpha_t-Q)}
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
&J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \\
&\frac{S_{\nu_1}(Q+\tau-\alpha_1)S_{\mathrm{K}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2}(\tau+\alpha_1)S_{\nu_3}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1+1}(Q+\tau+\alpha_4-\alpha_t)S_K(\tau+\alpha_4+\alpha_t)S_{\mu_2+1}(Q+\tau+\alpha_2-\alpha_s)S_{\mu_3+1}(\tau+\alpha_2+\alpha_s)}\\
+\ &\eta\frac{S_{\nu_1+1}(Q+\tau-\alpha_1)S_{\mathrm{K}+1}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2+1}(\tau+\alpha_1)S_{\nu_3+1}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1}(Q+\tau+\alpha_4-\alpha_t)S_{K+1}(\tau+\alpha_4+\alpha_t)S_{\mu_2}(Q+\tau+\alpha_2-\alpha_s)S_{\mu_3}(\tau+\alpha_2+\alpha_s)}
\end{aligned} \tag{4.71}$$

where $\eta = (-1)^{(1+\sum_i(\nu_i+\mu_i))/2}$.

We change here notations for the capital Latin letters denoting different spin structures. This is done to keep parametrization for the capital Latin letters in the formula for structure constants (4.53). Alternatively we could keep the same parametrization in formula for fusing matrix and change the notations in formula for structure constants.

Consider the limit

$$\alpha_t = \epsilon \to 0, \qquad \alpha_3 = \alpha_2, \qquad \alpha_4 = \alpha_1 \tag{4.72}$$

In this limit using formulas in appendix and (4.53) we have for the factor in front of integral

$$\mathcal{R}^{\mathcal{I}} \to \frac{2}{\pi\epsilon^2 C_{\mathcal{I}}(\alpha_s,\alpha_2,\alpha_1)} \frac{W_{NS}(0)W_E(Q-\alpha_2)W_L(Q-\alpha_s)}{W_F(\alpha_1)} \times \tag{4.73}$$
$$\frac{S_{\mathrm{F}}(2\alpha_1)}{S_B(Q-\alpha_s-\alpha_2+\alpha_1)S_D(\alpha_s+\alpha_1-\alpha_2)S_{\mathrm{E}}(2\alpha_2)S_{\mathrm{NS}}(\epsilon)}$$

Consider now the limit of the integrand (4.71).

In the limit (4.72) the arguments of S_K's in numerator and denominator coincide and they get canceled.

For the rest of S's in this limit we get for a_i in the argument of $S_{\nu_i}(\tau + a_i)$ and b_i in the argument of $S_{\mu_i+1}(-\tau + b_i)$:

$$\begin{aligned} a_1 &= Q-\alpha_1 & b_1 &= -\alpha_1 \\ a_2 &= \alpha_1 & b_2 &= \alpha_s-\alpha_2 \\ a_3 &= 2\alpha_2+\alpha_1-Q & b_3 &= Q-\alpha_2-\alpha_s \end{aligned} \tag{4.74}$$

From (4.74) we easily obtain:

$$\begin{aligned} a_1+b_1 &= Q-2\alpha_1 \\ a_1+b_2 &= Q-\alpha_1+\alpha_s-\alpha_2 \\ a_1+b_3 &= 2Q-\alpha_1-\alpha_s-\alpha_2 \end{aligned} \tag{4.75}$$

$$\begin{aligned} a_2+b_1 &= \epsilon \\ a_2+b_2 &= \alpha_1+\alpha_s-\alpha_2 \\ a_2+b_3 &= Q-\alpha_2-\alpha_s+\alpha_1 \end{aligned} \tag{4.76}$$

$$a_3 + b_1 = 2\alpha_2 - Q \tag{4.77}$$

$$a_3 + b_2 = \alpha_2 + \alpha_1 + \alpha_s - Q$$

$$a_3 + b_3 = \alpha_2 + \alpha_1 - \alpha_s$$

Note that

$$a_1 + b_3 = Q - (a_3 + b_2) \tag{4.78}$$

$$a_1 + b_2 = Q - (a_3 + b_3)$$

and

$$\sum_i (a_i + b_i) = Q \tag{4.79}$$

Assume that

$$\nu_1 + \mu_3 = \nu_3 + \mu_2 \mod 2 \tag{4.80}$$

$$\nu_1 + \mu_2 = \nu_3 + \mu_3 \mod 2$$

$$\nu_2 + \mu_1 = 1 \mod 2$$

Under these conditions we get from the theorem (4.56) , using formulas (4.75)-(4.80)

$$\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = \frac{2S_{\nu_2+\mu_2}(\alpha_1+\alpha_s-\alpha_2)S_{\nu_3+\mu_1}(2\alpha_2-Q)S_{\mathrm{NS}}(\epsilon)}{S_{\nu_1+\mu_1}(2\alpha_1)S_{\nu_2+\mu_3}(\alpha_2+\alpha_s-\alpha_1)} \tag{4.81}$$

Requiring additionally that

$$\nu_2 + \mu_3 = B \tag{4.82}$$

$$\nu_2 + \mu_2 = D$$

$$\nu_3 + \mu_1 = E$$

$$\nu_1 + \mu_1 = F$$

where these equalities as before understood in a sense, that odd sums identified with the NS sector, and even sums identified with the Ramond sectors, we get

$$\tilde{F}^{\mathcal{I}}_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix} = \lim_{\epsilon\to 0}\epsilon^2 F^{\mathcal{I}}_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix} = \frac{4}{\pi C_{\mathcal{I}}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_L(Q-\alpha_s)}{W_F(\alpha_1)W_E(\alpha_2)} \quad (4.83)$$

4.3 NS sector fusion matrix

Recall that structure constants in the NS sector are given by eq. (4.38) and (4.39) and fusion matrix by (4.45).

Remember that $NS = 1, \bmod 2$ and $R = 0, \bmod 2$. Putting $A = B = C = D = L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = \mu_3 = 0$, and using (4.68), we obtain for the $(i = 1, j = 1)$ component of the NS sector fusing matrices in the limit (4.54)

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}^1_1 = C_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)} \quad (4.84)$$

Putting $A = B = C = D = R$, $L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 0$, and using (4.68), we obtain for the $(i = 2, j = 1)$ component of the NS sector fusing matrices in the limit (4.54)

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}^2_1 = \tilde{C}_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)} \quad (4.85)$$

It is obvious to see that both choices of the ν_i and μ_i satisfy the conditions (4.65), (4.57), (4.67).

Putting $A = B = C = D = L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = \mu_3 = 0$, and using (4.83), we obtain for the $(i = 1, j = 1)$ component of the NS fusing matrices in the limit

(4.72)

$$\tilde{F}_{\alpha_s,0}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{1} = \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{1} = \frac{4}{\pi C_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)} \tag{4.86}$$

Putting $A = B = C = D = R$, $L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = 0$, $\mu_2 = \mu_3 = 1$, and using (4.83), we obtain for the $(i = 1, j = 2)$ component of the NS fusing matrix in the limit (4.72)

$$\tilde{F}_{\alpha_s,0}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{2} = \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{2} = \frac{4}{\pi \tilde{C}_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)} \tag{4.87}$$

It is again obvious to see that both sets of the values of ν_i and μ_i satisfy the conditions (4.57), (4.80) and (4.82).

Note also the relations:

$$F_{0,\alpha_s}\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}^{1}_{1} \tilde{F}_{\alpha_s,0}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{1} = \frac{S(0)S(\alpha_s)}{\pi^2 S(\alpha_1)S(\alpha_2)} \tag{4.88}$$

$$F_{0,\alpha_s}\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{bmatrix}^{2}_{1} \tilde{F}_{\alpha_s,0}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^{1}_{2} = \frac{S(0)S(\alpha_s)}{\pi^2 S(\alpha_1)S(\alpha_2)} \tag{4.89}$$

where $S(\alpha) = \sin \pi b(\alpha - Q/2) \sin \pi \frac{1}{b}(\alpha - Q/2)$.

We see that the relations (4.84)-(4.89) indeed have the structure of the equations (2),(4 and (5.

4.4 Fusion matrix in the Ramond sector

The fusion matrix in the Ramond sector unfortunately is not known in general. Although for some attempts see [73]. But for the degenerate primaries (4.21) fusion matrix can be computed via direct solutions of the corresponding differential equation for conformal blocks. In particular the necessary elements of the fusion matrix when one of the entries is the simplest degenerate field $R_{-b/2}$ are computed in [79, 80]. The degenerate field $R_{-b/2}$ possesses the OPE:

$$N_\alpha R_{-b/2} = C^{R_{\alpha-b/2}}_{N_\alpha R_{-b/2}} R_{\alpha-b/2} + C^{R_{\alpha+b/2}}_{N_\alpha R_{-b/2}} R_{\alpha+b/2} \tag{4.90}$$

$$R_\alpha R_{-b/2} = C^{N_{\alpha-b/2}}_{R_\alpha R_{-b/2}} N_{\alpha-b/2} + C^{N_{\alpha+b/2}}_{R_\alpha R_{-b/2}} N_{\alpha+b/2} \tag{4.91}$$

The corresponding structure constant can be computed in the Coulomb gas formalism using the screening integrals:

$$C^{R_{\alpha-b/2}}_{N_\alpha R_{-b/2}} = 1 \tag{4.92}$$

$$C^{R_{\alpha+b/2}}_{N_\alpha R_{-b/2}} = \pi\mu b^2 \gamma(bQ/2)\gamma(1-b\alpha)\gamma(b\alpha - bQ/2) = \frac{\mathcal{G}_{NS}(\alpha)}{\mathcal{G}_R(\alpha+b/2)} \tag{4.93}$$

$$C^{N_{\alpha-b/2}}_{R_\alpha R_{-b/2}} = 1 \tag{4.94}$$

$$C^{N_{\alpha+b/2}}_{R_\alpha R_{-b/2}} = 2i\pi\mu b^2 \gamma(bQ/2)\gamma(1/2-b\alpha)\gamma(b\alpha - b^2/2) = 2i\frac{\mathcal{G}_R(\alpha)}{\mathcal{G}_{NS}(\alpha+b/2)} \tag{4.95}$$

The fusion matrices can be computed having explicit expression of the conformal blocks with degenerate entries:

$$F_{R_{\alpha-b/2},0}\begin{bmatrix} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{bmatrix} = \frac{\Gamma(\alpha b - b^2/2 + 1/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2)\Gamma(1/2 - b^2/2)} \tag{4.96}$$

$$F_{R_{\alpha+b/2},0}\begin{bmatrix} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{bmatrix} = \frac{\Gamma(-\alpha b + b^2/2 + 3/2)\Gamma(-b^2)}{\Gamma(1-\alpha b)\Gamma(1/2 - b^2/2)} \tag{4.97}$$

$$F_{N_{\alpha-b/2},0}\begin{bmatrix} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{bmatrix} = \frac{\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2 - 1/2)\Gamma(1/2 - b^2/2)} \tag{4.98}$$

$$F_{N_{\alpha+b/2},0}\begin{bmatrix} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{bmatrix} = \frac{\Gamma(-\alpha b + b^2/2 + 1)\Gamma(-b^2)}{2i\Gamma(1/2-\alpha b)\Gamma(1/2 - b^2/2)} \tag{4.99}$$

It is an easy exercise to check that the values of the structure constants (4.92)-(4.95) and fusion matrices (4.96)-(4.99) satisfy the relations

$$C^{R_{\alpha-b/2}}_{N_\alpha R_{-b/2}} F_{R_{\alpha-b/2},0} \begin{bmatrix} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{bmatrix} = \frac{\Gamma(\alpha b - b^2/2+1/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2)\Gamma(1/2-b^2/2)} = \frac{W_{NS}(0)W_R(\alpha-b/2)}{W_{NS}(\alpha)W_R(-b/2)} \tag{4.100}$$

$$C^{R_{\alpha+b/2}}_{N_\alpha R_{-b/2}} F_{R_{\alpha+b/2},0} \begin{bmatrix} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{bmatrix} = \frac{\pi\mu b^2\gamma(bQ/2)\Gamma(-b^2)\Gamma(\alpha b - b^2/2-1/2)}{\Gamma(1/2-b^2/2)\Gamma(\alpha b)} = \frac{W_{NS}(0)W_R(\alpha+b/2)}{W_{NS}(\alpha)W_R(-b/2)} \tag{4.101}$$

$$C^{N_{\alpha-b/2}}_{R_\alpha R_{-b/2}} F_{N_{\alpha-b/2},0} \begin{bmatrix} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{bmatrix} = \frac{\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2-1/2)\Gamma(1/2-b^2/2)} = \frac{W_{NS}(0)W_{NS}(\alpha-b/2)}{W_R(\alpha)W_R(-b/2)} \tag{4.102}$$

$$C^{N_{\alpha+b/2}}_{R_\alpha R_{-b/2}} F_{N_{\alpha+b/2},0} \begin{bmatrix} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{bmatrix} = \frac{\pi\mu b^2\gamma(bQ/2)\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b+1/2)\Gamma(1/2-b^2/2)} = \frac{W_{NS}(0)W_{NS}(\alpha+b/2)}{W_R(\alpha)W_R(-b/2)} \tag{4.103}$$

One expects that similar relations should hold also for general expressions of the corresponding elements of fusion matrix in the RR sector. For example the fusions matrix with four RR entries should satisfy the relations

$$\lim_{\epsilon\to 0} F_{N_{\alpha_s},N_\epsilon} \begin{bmatrix} R_{\alpha_2} & R_{\alpha_2} \\ R_{\alpha_1} & R_{\alpha_1} \end{bmatrix} = \frac{4}{\pi\epsilon^2(C_R(\alpha_s|\alpha_2,\alpha_1)+\tilde{C}_R(\alpha_s|\alpha_1,\alpha_2))} \frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_R(\alpha_1)W_R(\alpha_2)} \tag{4.104}$$

$$F_{0,N_{\alpha_t}} \begin{bmatrix} R_{\alpha_3} & R_{\alpha_1} \\ R_{\alpha_3} & R_{\alpha_1} \end{bmatrix} = (C_R(\alpha_t|\alpha_1,\alpha_3)+\tilde{C}_R(\alpha_t|\alpha_1,\alpha_3)) \frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_R(Q-\alpha_1)W_R(Q-\alpha_3)} \tag{4.105}$$

One can hope that constraints like (4.104) and (4.105) may help to obtain the general expressions for the corresponding elements of the fusion matrix.

4.5 Defects in Super-Liouville theory

Two-point functions with a defect X insertion can be written as

$$\langle \Phi_i(z_1,\bar{z}_1) X \Phi_i(z_2,\bar{z}_2)\rangle = \frac{D^i}{(z_1-z_2)^{2\Delta_i}(\bar{z}_1-\bar{z}_2)^{2\Delta_i}}\,, \tag{4.106}$$

where

$$D^i = \mathcal{D}^i C_{ii} \tag{4.107}$$

and C_{ii} is a two-point function. They satisfy the Cardy-Lewellen equation for defects [56, 76, 82, 100]

$$\sum_k D^0 D^k \left(C_{ij}^k F_{k0} \begin{bmatrix} j & j \\ i & i \end{bmatrix} \right)^2 = D^i D^j\,. \tag{4.108}$$

Denote

$$D_{NS}(\alpha) = \langle N_\alpha X N_\alpha \rangle \tag{4.109}$$

$$D_R(\alpha) = \langle R_\alpha X R_\alpha \rangle \tag{4.110}$$

Let us take $j = R_{-b/2}$. Using (4.90), (4.91) and (4.100)-(4.103) one can obtain:

$$\Psi_{NS}(\alpha)\Psi_R(-b/2) = \Psi_R(\alpha-b/2) + \Psi_R(\alpha+b/2)\,, \tag{4.111}$$

$$\Psi_R(\alpha)\Psi_R(-b/2) = \Psi_{NS}(\alpha-b/2) + \Psi_{NS}(\alpha+b/2)\,, \tag{4.112}$$

where

$$\frac{D_{NS}(\alpha)}{D_{NS}(0)} = \Psi_{NS}(\alpha)\left(\frac{W_{NS}(0)}{W_{NS}(\alpha)}\right)^2\,, \tag{4.113}$$

$$\frac{D_R(\alpha)}{D_{NS}(0)} = \Psi_R(\alpha)\left(\frac{W_{NS}(0)}{W_R(\alpha)}\right)^2 . \tag{4.114}$$

The solution of the equations (4.111) and (4.112) is

$$\Psi_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2))\sin(\pi n b(\alpha - Q/2))}{\sin(\pi\frac{mb^{-1}Q}{2})\sin(\frac{\pi nbQ}{2})}, \tag{4.115}$$

$$\Psi_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2)))\sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{\sin(\frac{\pi mb^{-1}Q}{2})\sin(\frac{\pi nbQ}{2})}, \tag{4.116}$$

with $m - n$ is even.

Substituting (4.115) and (4.116) in (4.113) and (4.114) we obtain

$$D_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2))\sin(\pi n b(\alpha - Q/2))}{W_{NS}(\alpha)^2} \tag{4.117}$$

$$D_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2)))\sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{W_R(\alpha)^2} \tag{4.118}$$

Dividing by two-point functions (4.17) and (4.18) we obtain

$$\mathcal{D}_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2))\sin(\pi n b(\alpha - Q/2))}{\sin(\pi b^{-1}(\alpha - Q/2))\sin(\pi b(\alpha - Q/2))} \tag{4.119}$$

$$\mathcal{D}_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2)))\sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{\cos(\pi b^{-1}(\alpha - Q/2))\cos(\pi b(\alpha - Q/2))} \tag{4.120}$$

To obtain the continuous family of defects we use the strategy developed in [88, 96]. Namely consider $D_R(-b/2)$ as a parameter characterizing a defect. More precisely we define

$$A = \frac{D_R(-b/2)}{D_{NS}(0)}\left(\frac{W_R(-b/2)}{W_{NS}(0)}\right)^2 \tag{4.121}$$

Denoting also

$$D_{NS}(\alpha) = \frac{\tilde{\Psi}_{NS}(\alpha)}{W_{NS}(\alpha)^2}, \tag{4.122}$$

$$D_R(\alpha) = \frac{\tilde{\Psi}_R(\alpha)}{W_R(\alpha)^2}. \tag{4.123}$$

we obtain

$$A\tilde{\Psi}_{NS}(\alpha) = \tilde{\Psi}_R(\alpha - b/2) + \tilde{\Psi}_R(\alpha + b/2)\,, \tag{4.124}$$

$$A\tilde{\Psi}_{R}(\alpha) = \tilde{\Psi}_{NS}(\alpha - b/2) + \tilde{\Psi}_{NS}(\alpha + b/2)\,, \tag{4.125}$$

The solution of (4.124) and (4.125) is given by

$$\tilde{\Psi}_{NS}(\alpha; u) = \cosh(\pi(2\alpha - Q)u) \tag{4.126}$$

$$\tilde{\Psi}_{R}(\alpha; u) = \cosh(\pi(2\alpha - Q)u) \tag{4.127}$$

with a parameter u related to A by

$$2\cosh 2\pi b u = A\,. \tag{4.128}$$

Substituting (4.126) and (4.127) in (4.122) and (4.123) we obtain

$$D_{NS}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{W_{NS}(\alpha)^2} \tag{4.129}$$

$$D_{R}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{W_{R}(\alpha)^2} \tag{4.130}$$

Dividing by two-point functions (4.17) and (4.18) we obtain

$$\mathcal{D}_{NS}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{\sin(\pi b^{-1}(\alpha - Q/2))\sin(\pi b(\alpha - Q/2))} \tag{4.131}$$

$$\mathcal{D}_{R}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{\cos(\pi b^{-1}(\alpha - Q/2))\cos(\pi b(\alpha - Q/2))} \tag{4.132}$$

Chapter 5

The light asymptotic limit of Conformal blocks in Toda field theory

5.1 The light asymptotic limit of the Nekrasov partition functions

5.1.1 The Nekrasov partition functions of $\mathcal{N} = 2$ SYM theory

Consider $\mathcal{N} = 2$ SYM theory with gauge group $U(n)$ and $2n$ fundamental (more precisely n fundamental plus n anti-fundamental) hypermultiplets in Ω-background. The instanton part of the partition of this theory can be represented as

$$Z_{inst} = \sum_{\vec{Y}} F_{\vec{Y}} z^{|\vec{Y}|}, \tag{5.1}$$

where $\vec{Y}$ is an array of n Young diagrams, $|\vec{Y}|$ is the total number of boxes and z is the instanton counting parameter related to the gauge coupling in a standard manner. The coefficients $F_{\vec{Y}}$ are given by

$$F_{\vec{Y}} = \prod_{u=1}^{n} \prod_{v=1}^{n} \frac{Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v) Z_{bf}(a_u^{(1)}, Y_u \mid a_v^{(2)}, \varnothing)}{Z_{bf}(a_u^{(1)}, Y_u \mid a_v^{(1)}, Y_v)}, \tag{5.2}$$

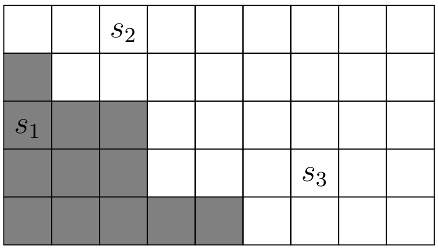

Figure 5.1: Arm and leg length with respect to a Young diagram (pictured in gray): $A(s_1) = 1$, $L(s_1) = 2$, $A(s_2) = -2$, $L(s_2) = -3$, $A(s_3) = -2$, $L(s_3) = -4$.

where

$$Z_{bf}(a, \lambda \mid b, \mu) = \tag{5.3}$$

$$\prod_{s\in\lambda} (a - b - \epsilon_1 L_\mu(s) + \epsilon_2(1 + A_\lambda(s))) \prod_{s\in\mu} (a - b + \epsilon_1(1 + L_\lambda(s)) - \epsilon_2 A_\mu(s)) \,.$$

Here $A_\lambda(s)$ and $L_\lambda(s)$ are correspondingly the arm-length and leg-length of the square s towards the Young tableau λ, defined as oriented vertical and horizontal distances of the square s to outer boundary of the Young tableau λ (see Fig.5.1).

Let us clarify our conventions on gauge theory parameters $a_u^{(0,1,2)}$, $u = 1, 2, \ldots, n$. The parameters $a_u^{(1)}$ are expectation values of the scalar field in vector multiplet. Without loss of generality we'll assume that the "center of mass" of these expectation values is zero

$$\bar{a}^{(1)} = \frac{1}{n} \sum_{u=1}^{n} a_u^{(1)} = 0 \,. \tag{5.4}$$

In fact this is not a loss of generality since a nonzero center of mass can be absorbed by shifting hypermultiplet masses. Furthermore $a_u^{(0)}$ ($a_u^{(2)}$) are the masses of fundamental (anti-fundamental) hypers. Finally the ϵ_1, ϵ_2 are the Ω-background parameters. Sometimes we will use the notation $\epsilon = \epsilon_1 + \epsilon_2$.

Due to AGT duality, this partition function is directly related to specific four point conformal block in 2d A_{n-1} Toda field theory. Before describing this relation let us briefly recall few facts about Toda theory.

5.1.2 Preliminaries on A_{n-1} Toda CFT and AGT relation

These are 2d CFT theories which besides the spin 2 holomorphic energy momentum $W^{(2)}(z) \equiv T(z)$ are endowed with additional higher spin $s = 3, 4, \ldots, n$ currents $W^{(3)}, \ldots, W^{(n)}$ with Virasoro central charge conventionally parameterised as

$$c = n - 1 + 12Q^2 \,,$$

where the vector "background charge"

$$Q = \rho(b + 1/b)$$

with ρ being the Weyl vector of the algebra A_{n-1} and b is the dimensionless coupling constant of Toda theory. In what follows it would be convenient to represent the roots, weights and Cartan elements of A_{n-1} as n-component vectors with the usual Kronecker scalar product, subject to the condition that sum of components is zero. Of course this is equivalent to more conventional representation of these quantities as diagonal traceless $n \times n$ matrices with the pairing given by trace. In this representation the Weyl vector is given by

$$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right) \quad \text{or} \quad \rho_u = \frac{n+1}{2} - u \tag{5.5}$$

and for the central charge we'll get

$$c = (n-1)(1 + n(n+1)q^2) \,,$$

where for the later use we have introduced the parameter

$$q = b + \frac{1}{b} \,.$$

For further reference let us quote here explicit expressions for the highest weight ω_1 of the first fundamental representation and for its complete set of weights $h_1, \ldots, h_n$ $(h_1 = \omega_1)$

$$
\begin{aligned}
(\omega_1)_k &= \delta_{1,k} - 1/n\,; \\
(h_l)_k &= \delta_{l,k} - 1/n\,.
\end{aligned}
\tag{5.6}
$$

The primary fields V_α (here we concentrate only on left moving holomorphic parts) are parameterized by vectors α with vanishing center of mass. Their conformal weights are given by

$$
h_\alpha = \frac{\alpha(2Q-\alpha)}{2}\,. \tag{5.7}
$$

In what follows a special role is played by the fields $V_{\lambda\omega_1}$ with the dimensions:

$$
h_{\lambda\omega_1} = \frac{\lambda(n-1)}{2}\left(q - \frac{\lambda}{n}\right)\,. \tag{5.8}
$$

A four point block:

$$
\langle V_{\alpha^{(4)}}(\infty) V_{\lambda^{(3)}\omega_1}(1) V_{\lambda^{(2)}\omega_1}(q) V_{\alpha^{(1)}}(0)\rangle_\alpha = q^{h_\alpha - h_{\alpha^{(1)}} - h_{\alpha^{(2)}\omega_1}} \mathcal{F}_\alpha \begin{bmatrix} \lambda^{(3)}\omega_1 & \lambda^{(2)}\omega_1 \\ \alpha^{(4)} & \alpha^{(1)} \end{bmatrix} (q)\,, \tag{5.9}
$$

where α specifies the W-family running in s-channel, is closely related to the gauge partition function Z_{inst} see (5.15) (AGT relation). First of all, the instanton counting parameter q gets identified with the cross ratio of insertion points in CFT block as it was already anticipated in (5.9) and the Toda parameter b is related to Ω-background parameters via

$$
b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,. \tag{5.10}
$$

The map between the gauge parameters in (5.1) and conformal block parameters in (5.9) should be established from the following rules (see Fig.6.2). To formulate them we define the

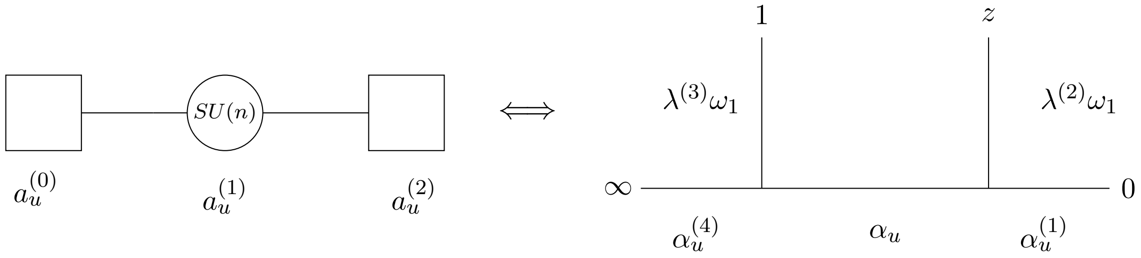

Figure 5.2: On the left: the quiver diagram for the conformal $U(n)$ gauge theory. On the right: the diagram of the conformal block for the dual Toda field theory.

rescaled gauge parameters

$$A_u^{(0)} = \frac{a_u^{(0)}}{\sqrt{\epsilon_1\epsilon_2}}\,; \qquad A_u^{(1)} = \frac{a_u^{(1)}}{\sqrt{\epsilon_1\epsilon_2}}\,; \qquad A_u^{(2)} = \frac{a_u^{(2)}}{\sqrt{\epsilon_1\epsilon_2}}\,. \tag{5.11}$$

- The differences between the "centers of mases" of the successive rescaled gauge parameters (6.26) give the charges of the "vertical" entries of the conformal block:

$$\bar{A}^{(1)} - \bar{A}^{(0)} = \frac{\lambda^{(3)}}{n}\,; \qquad \bar{A}^{(2)} - \bar{A}^{(1)} = \frac{\lambda^{(2)}}{n}\,. \tag{5.12}$$

- The rescaled gauge parameters with the subtracted centers of masses give the momenta of the "horizontal" entries of the conformal block:

$$\begin{aligned} A_u^{(0)} - \bar{A}^{(0)} &= Q_u - \alpha_u^{(4)}\,; \\ A_u^{(1)} - \bar{A}^{(1)} &= Q_u - \alpha_u\,; \\ A_u^{(2)} - \bar{A}^{(2)} &= Q_u - \alpha_u^{(1)}\,. \end{aligned} \tag{5.13}$$

Using (6.2), (5.5) and (6.26)-(6.28) we obtain the relation between the gauge and conformal parameters:

$$\begin{aligned} \frac{a_u^{(0)}}{\sqrt{\epsilon_1\epsilon_2}} &= -\alpha_u^{(4)} - \frac{\lambda^{(3)}}{n} + q\left(\frac{n+1}{2} - u\right); \\ \frac{a_u^{(1)}}{\sqrt{\epsilon_1\epsilon_2}} &= -\alpha_u + q\left(\frac{n+1}{2} - u\right); \end{aligned}$$

$$\frac{a_u^{(2)}}{\sqrt{\epsilon_1\epsilon_2}} = -\alpha_u^{(1)} + \frac{\lambda^{(2)}}{n} + q\left(\frac{n+1}{2} - u\right). \tag{5.14}$$

With all these preparations one can write the AGT correspondence between the Nekrasov function defined in (5.1) and the conformal block in (5.9) (see [102, 104]):

$$Z_{inst} = (1-z)^{\lambda^{(3)}\left(q-\frac{\lambda^{(2)}}{n}\right)} \mathcal{F}_\alpha \begin{bmatrix} \lambda^{(3)}\omega_1 & \lambda^{(2)}\omega_1 \\ \alpha^{(4)} & \alpha^{(1)} \end{bmatrix} (z). \tag{5.15}$$

5.1.3 Light asymptotic limit

In this paper we are interested in so called "light" asymptotic limit i.e. the central charge is sent to infinity (i.e. $b \to 0$) while keeping the dimensions finite. It follows from (5.7) that to reach this limit one can simply put

$$\alpha_u^{(1)} = b\eta_u^{(1)}; \qquad \alpha_u^{(4)} = b\eta_u^{(4)}; \qquad \alpha_u = b\eta_u \tag{5.16}$$

keeping all the parameters η finite. As for the parameters λ of the special fields $V_{\lambda\omega_1}$, there are two inequivalent alternatives:

(i) $\lambda = b\eta$

or

(ii) $nq - \lambda = b\eta$.

Though in both cases the conformal dimension takes the same value (see eq. (5.8))

$$h = \frac{\eta(n-1)}{2},$$

these fields are not identical, which can be seen e.g. from the fact that the zero mode eigenvalues of odd W-currents for these fields have the same absolute values but opposite signs. In fact the fields $V_{\lambda\omega_1}$ and $V_{(nq-\lambda)\omega_1}$ can be considered as conjugate to each other in the usual sense, since their two point function is non-zero. It is easy to check that $V_{(nq-\lambda)\omega_1}$ is equivalent to $V_{\lambda\omega_{n-1}}$

(ω_{n-1} is the highest weight of the anti-fundamental representation) since the corresponding momentum parameters $Q - \lambda\omega_1$ and $Q - (nq - \lambda)\omega_{n-1}$ are related by a Weyl transformation.

In this paper we will investigate in great detail the case when $V_{\lambda_3\omega_1}$ is a light field of type (i) while $V_{\lambda_2\omega_1}$ is of type (ii). In other words we set

$$\lambda^{(3)} = b\eta^{(3)}; \qquad nq - \lambda^{(2)} = b\eta^{(2)}\,. \tag{5.17}$$

For such choice we will see below, that the corresponding instanton sum simplifies drastically and leads to a simple explicit expression for the conformal block. Note that this choice is very convenient since the prefactor in front of conformal block in (5.15) now goes to 1 in the light asymptotic limit. The opposite case when two special fields are of the same type, has been investigated in [115] in particular case of A_2 Toda. In the case considered in [115] the above mentioned prefactor survives.

Coming back to our case of interest using (5.17), (6.30) we can rewrite the AGT map (6.29) as

$$\begin{aligned} a_u^{(0)} &= -\epsilon_1\left(\eta_u^{(4)} + \frac{\eta^{(3)}}{n}\right) + \epsilon\left(\frac{n+1}{2} - u\right); \\ a_u^{(1)} &= -\epsilon_1\eta_u + \epsilon\left(\frac{n+1}{2} - u\right); \\ a_u^{(2)} &= -\epsilon_1\left(\eta_u^{(1)} + \frac{\eta^{(2)}}{n}\right) + \epsilon\left(\frac{n+3}{2} - u\right). \end{aligned} \tag{5.18}$$

In view of (6.25) the small b limit is equivalent to $\epsilon_1 \to 0$. Hence we are interested in the $\epsilon_1 \to 0$ limit of (6.3). We will see that the degree of ϵ_1 (denote it by N) is non-negative for arbitrary array of Young diagrams Y_v and that the degree $N = 0$ (hence a finite non-zero limit exists) if and only if each Young diagram Y_v ($v = 1, 2, \ldots, n$) has at most $v - 1$ rows.
From (6.3) we see that

$$N = n_1 + n_2 - n_3 \tag{5.19}$$

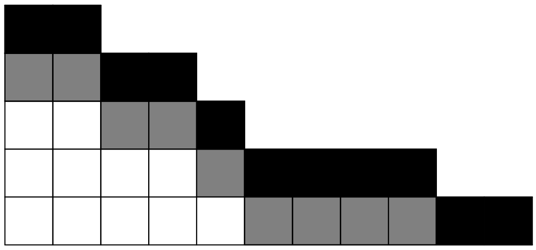

Figure 5.3: This picture shows that there are $Y_{v,1}$ boxes such that $A_{Y_v} = 0$ (painted black) and $Y_{v,2}$ boxes with $A_{Y_v} = 1$ (painted grey).

with n_1, n_2 being the ϵ_1 degrees of the first and second factors in the numerator of (6.3) respectively and n_3 is the ϵ_1 degree of its denominator.

Let us derive n_1, using (6.4) for $Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v)$ and inserting (5.18) we'll get

$$Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v) = \tag{5.20}$$
$$\prod_{s\in Y_v} \left(\epsilon_1(1 + L_\varnothing(s) + \eta_v - \eta_u^{(4)} - \tfrac{\eta^{(3)}}{n}) + \epsilon(v-u) - \epsilon_2 A_{Y_v}(s)\right).$$

A factor in (5.20) contributes to the degree of ϵ_1 if its part proportional to ϵ_2 vanishes. Evidently this happens when $A_{Y_v}(s) = v - u$. Since the box $s \in Y_v$, $A_{Y_v}(s) \geq 0$, we see that when $v = 1$ the only admissible value for u is $u = 1$. It is obvious from Fig. 5.3 that there are exactly $Y_{1,1}$ boxes in Y_1 for which the arm-length vanishes (here and below we denote by $Y_{v,i}$ the number of boxes in the i'th row of diagram Y_v). When $v = 2$, there are two admissible values $u = 1$ or $u = 2$. As in the previous case the number of the boxes with zero arm-length (case $u = 2$) is equal $Y_{2,1}$. Similarly, a simple inspection shows that the number of boxes with unit arm-lengths (case $u = 2$) are equal to $Y_{2,2}$. This analysis can be easily continued for other values of v with result summarized in the table below

	u=1	u=2	u=3	...	u=n
v=1	$Y_{1,1}$				
v=2	$Y_{2,2}$	$Y_{2,1}$			
v=3	$Y_{3,3}$	$Y_{3,2}$	$Y_{3,1}$		
...				...	
v=n	$Y_{n,n}$	$Y_{n,n-1}$	$Y_{n,n-2}$	...	$Y_{n,1}$

Obviously the degree n_1 is nothing but the sum of all entries of this table.

$$n_1 = \sum_{u=1}^{n} \sum_{k=1}^{u} Y_{u,k} \,. \tag{5.21}$$

With almost identical arguments it is possible to show that $n_2 = n_1$. Finally, an analogous consideration for the degree n_3 gives

$$n_3 = \sum_{u=1}^{n} \sum_{k=1}^{u} Y_{u,k} + \sum_{u=1}^{n} \sum_{k=1}^{u-1} Y_{u,k} \,. \tag{5.22}$$

Thus for the total degree (5.19) we get

$$N = \sum_{u=1}^{n} Y_{u,u} \,. \tag{5.23}$$

Each term here is non-negative and in order to get a vanishing total degree $N = 0$, the array of Young diagrams should satisfy the conditions $Y_{1,1} = Y_{2,2} = \cdots Y_{n,n} = 0$, which means that each Young diagram Y_u consists of at most $u-1$ rows.

5.1.4 Nekrasov partition function of $\mathcal{N} = 2$ SYM theory in the light asymptotic limit

Now our purpose is to derive $F_{\vec{Y}}$ explicitly in the light asymptotic limit. To do this let us study the first factor in the numerator of (6.3) which, according to (6.4) and (5.18), is given by

$$Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v) = \tag{5.24}$$
$$\textstyle\prod_{s \in Y_v} \left(\epsilon_1(1 + L_{\varnothing}(s) + \eta_v - \eta_u^{(4)} - \frac{\eta^{(3)}}{n}) + \epsilon(v-u) - \epsilon_2 A_{Y_v}(s)\right).$$

Let $Y_v^{(1)}$ be the set of such boxes s of the Young diagram Y_v (with at most $v-1$ rows) that the coefficient of ϵ_2 vanishes in the respective factor of (5.24)), i.e.

$$v - u - A_{Y_v}(s) = 0 \,. \tag{5.25}$$

This can happen only when $i \equiv v-u \geq 0$. Thus for the part of (5.24) under discussion we get

$$Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v^{(1)}) = \prod_{s\in Y_v^{(1)}} \epsilon_1(1+L_\varnothing(s)+\eta_v-\eta_u^{(4)}-\tfrac{\eta^{(3)}}{n}+v-u)\,. \tag{5.26}$$

We have already seen in previous chapter that there are exactly $Y_{v,i+1}$ boxes, satisfying (5.25). These boxes are distributed in Y_v in such a way that there is a single box on j-th column (denote it by s_j) for each $j=1,\ldots,Y_{v,i+1}$ (see Fig.5.3). Taking into account that $L_\varnothing(s_j)=-j$, we can rewrite (5.26) as

$$Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v^{(1)}) = \prod_{j=1}^{Y_{v,i+1}} \epsilon_1(\eta_v-\eta_{v-i}^{(4)}-\tfrac{\eta^{(3)}}{n}+1-j+i). \tag{5.27}$$

Now let's look on the alternative case of the set $Y_v^{(2)}$ of those boxes which do not satisfy (5.25) so that in the related factors we can safely set $\epsilon_1=0$. Again from (5.20) we'll get

$$Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v^{(2)}) = \prod_{s\in Y_v^{(2)}} \epsilon_2(v-u-A_{Y_v}(s)). \tag{5.28}$$

Carefully examining the cases $v-u-A_{Y_v}(s)>0$ and $v-u-A_{Y_v}(s)<0$ separately we get

$$\begin{gathered}\prod_{u=1}^n\prod_{v=1}^n Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v^{(2)}) = \\ \prod_{v=2}^n\prod_{i=1}^{v-1}\left((-)^{n-v-1-i}(n-v-1-i)!(v-i)!\epsilon_2^{(n-1)}\right)^{Y_{v,i}}.\end{gathered} \tag{5.29}$$

Combining (5.27) with (5.29) we obtain

$$\begin{gathered}\prod_{u=1}^n\prod_{v=1}^n Z_{bf}(a_u^{(0)}, \varnothing \mid a_v^{(1)}, Y_v) = \\ \prod_{v=2}^n\prod_{i=0}^{v-2}\prod_{j=1}^{Y_{v,i+1}} \epsilon_1(\eta_v-\eta_{v-i}^{(4)}-\tfrac{\eta^{(3)}}{n}+1-j+i)\times \\ \prod_{v=2}^n\prod_{i=1}^{v-1}\left((-)^{n-v-1-i}(n-v-1-i)!(v-i)!\epsilon_2^{(n-1)}\right)^{Y_{v,i}}.\end{gathered} \tag{5.30}$$

Similar arguments for the second factor in the denominator of (6.3) lead to the expression

$$\prod_{u=1}^{n}\prod_{v=1}^{n} Z_{bf}(a_u^{(1)}, Y_u \mid a_v^{(2)}, \varnothing) = \tag{5.31}$$
$$\prod_{u=2}^{n}\prod_{i=1}^{u-1}\left(\epsilon_2^{n-1}(-)^i i!(n-1-i)!\right)^{Y_{u,u-i}} \times$$
$$\prod_{u=2}^{n}\prod_{i=0}^{u-2}\prod_{j=1}^{Y_{u,i+1}} \epsilon_1(\eta_{u-i}^{(1)} - \eta_u + \tfrac{\eta^{(2)}}{n} + j - i - 1).$$

The derivation of the denominator of (6.3) though somewhat lengthier but still is quite straightforward and leads to

$$\prod_{u=1}^{n}\prod_{v=1}^{n} Z_{bf}(a_u^{(1)}Y_u \mid a_v^{(1)}Y_v) = \tag{5.32}$$
$$\prod_{l=1}^{n-1}\prod_{v=1}^{n-l}(-\epsilon_1)^{Y_{v+l,l}}\prod_{k=l}^{v+l-1}\prod_{i=1+Y_{v+l,k+1}}^{Y_{v+l,k}}(\eta_{v+l} - \eta_v + l + Y_{v,k-l+1} - i) \times$$
$$\prod_{l=0}^{n-2}\prod_{v=l+2}^{n}\epsilon_1^{Y_{v,l+1}}\prod_{k=l+1}^{v-1}\prod_{i=1+Y_{v,k+1}}^{Y_{v,k}}(\eta_v - \eta_{v-l} + l + 1 + Y_{v-l,k-l} - i) \times$$
$$\prod_{v=2}^{n}\prod_{i=1}^{v-1}(\epsilon_2^{n-1}(-)^{i-1}(i-1)!(n-i)!)^{Y_{v,v-i}}((-)^{n-v-1-i}(n-v-1-i)!(v-i)!\epsilon_2^{(n-1)})^{Y_{v,i}},$$

where the products on the second (third) line comes from the terms $u < v$ ($u > v$) and the last line results in from diagonal $u = v$ terms. Notice that, as we have already proved earlier, the order in ϵ_1 of the numerator and the denominator coincide safely providing a finite $\epsilon_1 \to 0$ limit. Also dependence of the ratio in ϵ_2 disappears (as it should from scaling arguments). Inserting (5.30), (5.31) and (5.32) in (6.3) for $F_{\vec{Y}}$ in the light asymptotic limit we finally get

$$F_{\vec{Y}} = \prod_{u=2}^{n}\prod_{v=2}^{u}\left(\frac{u-v+1}{n-u+v-1}\right)^{Y_{u,v-1}} \tag{5.33}$$
$$\frac{\prod_{i=0}^{Y_{u,u-v+1}-1}\left(-\eta_u+\eta_v^{(4)}+\frac{\eta^{(3)}}{n}-u+v+i\right)\left(\eta_u-\eta_v^{(1)}-\frac{\eta^{(2)}}{n}+u-v-i\right)}{\prod_{k=u-v+1}^{u-1}\prod_{i=Y_{u,k+1}}^{Y_{u,k}-1}(\eta_u-\eta_{v-1}+u-v+Y_{v-1,k+v-u}-i)(\eta_u-\eta_v+u-v+Y_{v,k+v-u}-i)}\ .$$

where $Y_{v,i}$ is the number of boxes in the i'th row of diagram Y_v. As already mentioned in the case under consideration the prefactor in (5.15) becomes 1, hence

$$\mathcal{F}_{CFT} = Z_{inst} = \sum_{\vec{Y}} F_{\vec{Y}} z^{|\vec{Y}|}, \tag{5.34}$$

The sum is taken over all Young diagrams Y_u, $u = 2, \ldots, n$, with at most $u-1$ rows, i.e. over all allowed row lengths $Y_{u,1} \geq Y_{u,2} \geq \ldots \geq Y_{u,u-1} \geq 0$.

Let us consider the particular cases when $n = 2$ (Liouville) and $n = 3$ separately.

When $n = 2$ we have a single sum

$$\begin{aligned} \mathcal{F}_{Liouv} &= \sum_{i=0}^{\infty} \frac{\left(\eta_2^{(4)}+\eta_1+\frac{\eta^{(3)}}{2}\right)_i \left(\eta_2^{(1)}+\eta_1+\frac{\eta^{(2)}}{2}\right)_i}{i!(2\eta_1)_i}\ z^i \\ &= {}_2F_1\left(\eta_2^{(1)} + \eta_1 + \frac{\eta^{(2)}}{2}\,, \eta_2^{(4)} + \eta_1 + \frac{\eta^{(3)}}{2}\,, 2\eta_1;\ z\right)\,, \end{aligned} \tag{5.35}$$

where ${}_2F_1(a, b; c; x)$ is the Gauss hyper-geometric function. This is a well known result in Liouville theory [114–117].

When $n = 3$ we get

$$\begin{aligned} \mathcal{F}_{W_3} &= \sum_{i,j,l=0}^{\infty} (-)^l 2^{j-i}\ z^{2l+i+j} \\ &\times \left(\frac{\eta^{(3)}}{3} - \eta_2 + \eta_2^{(4)}\right)_i \left(\frac{\eta^{(3)}}{3} - \eta_3 + \eta_2^{(4)} - 1\right)_l \left(\frac{\eta^{(3)}}{3} - \eta_3 + \eta_3^{(4)}\right)_{j+l} \\ &\times \frac{\left(\frac{\eta^{(2)}}{3}-\eta_2+\eta_2^{(1)}\right)_i \left(\frac{\eta^{(2)}}{3}-\eta_3+\eta_2^{(1)}-1\right)_l \left(\frac{\eta^{(2)}}{3}-\eta_3+\eta_3^{(1)}\right)_{j+l}}{i!j!l!(\eta_1-\eta_2)_i(\eta_1-\eta_3-1)_l(\eta_2-\eta_3)_l(\eta_2-\eta_3-i-1)_l(\eta_2-\eta_3+l-i)_j}\,, \end{aligned} \tag{5.36}$$

This formula completes the result of [115] where the light four-point function of W_3-theory has been computed in the case when both the second and the third insertions were light primaries of the same sort:

$$\lambda^{(3)} = b\eta^{(3)}; \qquad \lambda^{(2)} = b\eta^{(2)}\,, \tag{5.37}$$

whereas (5.36) is obtained with the choice specified in (5.17). In the next section we present an alternative calculation of (5.36) based on the integral representation of the conformal blocks in the light asymptotic limit used in [115].

5.2 Light asymptotic limit for the four point block in W_3

It has been shown in [115] that the multi-point conformal blocks of the W_3 theory in the light asymptotic limit can be constructed in the terms of $sl(3)$ three-point invariant functions. For the details we refer the reader to the original paper. Here we'll introduce the necessary notations and state the relevant results.

It is well known that the $sl(3)$ generators can be represented as operators acting on the triple of the isospin variables $Z = (w, x, y)$. To construct a multi-point block one should multiply several three-point functions then identify pairs of isospin variables corresponding to the internal states and integrate them out with an appropriate measure. At the end one specializes the external leg variables putting

$$Z = \left(\frac{1}{2}\,z^2, z, z\right) , \tag{5.38}$$

where z is the insertion point.

In particular the four-point block can be represented as

$$\mathcal{F} = \int_C d^3 Z_s \mathcal{E}_1(j_2, j_1, J_s^{\omega}|Z_2, Z_1, Z_s)\mathcal{E}_2(j_3, j_4^{\omega}, J_s^{*\omega}|Z_3, Z_4, Z_s) , \tag{5.39}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the appropriate three point invariants given by *

$$\begin{aligned}
\mathcal{E}_1(j_1, j_2, j_3|Z_1, Z_2, Z_3) &= \chi_{123}^{-J}\rho_{12}^{-J-r_2+s_3}\rho_{13}^{-J-r_3+s_2}\rho_{23}^{J-s_2}\rho_{32}^{J-s_3} ; \\
\mathcal{E}_2(j_1, j_2, j_3|Z_1, Z_2, Z_3) &= \sigma_{123}^{J}\rho_{21}^{J+r_3-s_2}\rho_{31}^{J+r_2-s_3}\rho_{23}^{-J-r_3}\rho_{32}^{-J-r_2}
\end{aligned} \tag{5.40}$$

*We have different three point invariants, since the second and third light fields are of different kinds as specified in (5.17). The case of fields of the same kind is analysed in [115].

with

$$\rho_{ij} = y_i\,(x_i - x_j) - (w_i - w_j)\;;$$

$$\sigma_{ijk} = x_i w_j - w_i x_j - x_i w_k + w_i x_k - w_j x_k + x_j w_k\;;$$

$$\chi_{ijk} = y_i w_j - w_i y_j + y_i y_j\,(x_i - x_j) - y_i w_k + w_i y_k + y_i y_k\,(x_k - x_i)$$
$$-w_j y_k + y_j w_k + y_j y_k\,(x_j - x_k)\,, \tag{5.41}$$

the quantities $j = (r, s)$, $j^* = (2 - r, 2 - s)$, $j^\omega = (s, r)$ (see [115]) specify the primary fields and are related to the charge vectors η_u introduced in section (6.3.2) as

$$r = \eta_1 - \eta_2\,; \qquad s = \eta_2 - \eta_3\,; \qquad \eta_1 + \eta_2 + \eta_3 = 0 \tag{5.42}$$

and, finally,

$$J = (h_2, j_1 + j_2 + j_3) = \frac{1}{3}\,(s_1 + s_2 + s_3 - r_1 - r_2 - r_3)\,. \tag{5.43}$$

Due to (5.42) and (6.30), (5.17) for our case we have

$$\begin{aligned} r_s &= \eta_1 - \eta_2\,; \qquad & s_s &= \eta_2 - \eta_3\,;\\ r_1 &= \eta_1^{(1)} - \eta_2^{(1)}\,; \qquad & s_1 &= \eta_2^{(1)} - \eta_3^{(1)}\,;\\ r_4 &= \eta_1^{(4)} - \eta_2^{(4)}\,; \qquad & s_4 &= \eta_2^{(4)} - \eta_3^{(4)}\,;\\ s_2 &= \eta^{(2)}\,; \qquad & r_2 &= 0\,;\\ r_3 &= \eta^{(3)}\,; \qquad & s_3 &= 0\,. \end{aligned} \tag{5.44}$$

As usual, using projective invariance we can specify the insertion points as $(z_4, z_3, z_2, z_1) \to (\infty, 1, x, 0)$, see Fig.6.2. Under this specification, after dropping out an unimportant constant (infinite) factor, $\mathcal{E}_2$ gets simplified

$$\mathcal{E}_2(j_3, j_4^{\omega}, J_s^{*\omega}|Z_3, Z_4, Z_s) = (1-x_s)^{\frac{1}{3}(r_4+s_s-r_s-r_3-s_4)}\, \rho_{s,3}^{\frac{r_4+s_s-r_s-r_3-s_4}{3}+s_4+r_s-2}\,. \tag{5.45}$$

Putting

$$x_2 \to z\,; \qquad y_2 \to z\,; \qquad w_2 \to \tfrac{z^2}{2}\,; \qquad x_1 \to 0\,;$$
$$y_1 \to 0\,; \qquad w_1 \to 0\,; \qquad x_3 \to 1\,; \qquad y_3 \to 1\,; \qquad w_3 \to \tfrac{1}{2}\,,$$

as instructed in (5.38) and dropping out the usual factor
$z^{h_{\alpha_s}-h_{\alpha^{(1)}}-h_{\lambda^{(2)}\omega_1}} = z^{r_s+s_s-(r_1+s_1)-s_2}$, up to an unimportant constant multiplier we get the integral

$$\begin{aligned}\mathcal{F} = \int dx_s\, dy_s\, dw_s \left(w_s - y_s\left(x_s - \frac{z}{2}\right)\right)^{\frac{1}{3}(r_1+s_s-s_1-s_2-r_s)} \\ \times w_s^{\frac{1}{3}(-r_1-s_s-2s_1+s_2+r_s)}\,(1-x_s)^{\frac{1}{3}(-r_3-s_4+s_s+r_4-r_s)}\,(w_s - x_s y_s)^{\frac{1}{3}(-r_1-s_s+s_1+s_2-2r_s)} \\ \times \left(w_s - z\left(x_s - \frac{z}{2}\right)\right)^{\frac{1}{3}(r_1-2s_s+2s_1-s_2-r_s)} \left(w_s - (x_s-1)\,y_s - \frac{1}{2}\right)^{\frac{1}{3}(-r_3+2s_4+s_s+r_4+2r_s)-2}.\end{aligned} \tag{5.46}$$

After the change of the variables

$$x_s \to \frac{x}{2w}\,; \qquad w_s \to \frac{1}{2w}\,; \qquad y_s \to \frac{y}{xy-w} \tag{5.47}$$

we'll get

$$\begin{aligned}\mathcal{F} &= \int_{\mathcal{C}} dx\, dy\, dw w^{\frac{1}{3}(r_3+s_4+2s_s-r_4+r_s)-2}(1-yz)^{\frac{1}{3}(r_1+s_s-s_1-s_2-r_s)} \\ &\times \left(w - \frac{x}{2}\right)^{\frac{1}{3}(-r_3-s_4+s_s+r_4-r_s)} \left(wz^2 - xz + 1\right)^{\frac{1}{3}(r_1-2s_s+2s_1-s_2-r_s)} \\ &\times (xy-w)^{\frac{1}{3}(r_3-2s_4-s_s-r_4+r_s)}(-w+(x-2)y+1)^{\frac{1}{3}(-r_3+2s_4+s_s+r_4+2r_s)-2}.\end{aligned} \tag{5.48}$$

Here is the result of the integration (for the details of the calculation see appendix:??)

$$\mathcal{F} = \sum_{m,n,k=0}^{\infty} \sum_{l=0}^{m} (-)^{k+l} 2^{n-m} z^{2k+m+n}$$
$$\times \left(\tfrac{1}{3}\left(s_s + 2r_s - r_3 - s_4 + r_4\right)\right)_l \left(\tfrac{1}{3}\left(-s_s + r_s - r_1 + s_1 + s_2\right)\right)_m \tag{5.49}$$
$$\times \frac{\left(\frac{1}{3}(2s_s+r_s-r_1-2s_1+s_2)\right)_{k+n}\left(\frac{1}{3}(2s_s+r_s+r_3-2s_4-r_4)\right)_{k+n}\left(\frac{1}{3}(2s_s+r_s+r_3+s_4-r_4-3)\right)_{k-l+m}}{k!l!n!(m-l)!(r_s)_l(s_s)_{k-l+n}(s_s+r_s-1)_{k+m}} .$$

Though this expression looks different from (5.36) below we argue that in fact they coincide. First we will prove this analytically up to the second order in z. It is convenient to rewrite (5.49) and (5.36) in terms of parameters A_1, A_2, B_1, B_2 defined as

$$A_1 = \frac{1}{3}\left(r - r_1 - s + s_1 + s_2\right); \qquad B_1 = \frac{1}{3}\left(r - r_1 + 2s - 2s_1 + s_2\right); \tag{5.50}$$
$$A_2 = \frac{1}{3}\left(r + r_3 + s_4 - s - r_4\right); \qquad B_2 = \frac{1}{3}\left(r + r_3 - 2s_4 + 2s - r_4\right). \tag{5.51}$$

For (5.49) we will get

$$\mathcal{F} = \sum_{k,n,m=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^{k+l}2^{n-m}(A_1)_m(r_s-A_2)_l(B_1)_{k+n}(B_2)_{k+n}(A_2+s_s-1)_{k-l+m}z^{2k+m+n}}{k!l!n!(m-l)!(r_s)_l(s_s)_{k-l+n}(r_s+s_s-1)_{k+m}} \tag{5.52}$$

and (5.36) is given

$$\mathcal{F}_{W_3} = \sum_{k,n,m=0}^{\infty} \frac{(-1)^k 2^{m-n}(A_1)_n(A_2)_n(A_1+s_s-1)_k(A_2+s_s-1)_k(B_1)_{k+m}(B_2)_{k+m}z^{2k+m+n}}{k!m!n!(s_s)_k(r_s)_n(-n+s_s-1)_k(r_s+s_s-1)_k(k-n+s_s)_m} \tag{5.53}$$

It is easy to see from (5.52) that the term proportional to z is

$$\mathcal{F}^{(1)} = \frac{A_1(A_2+s_s-1)}{2(r_s+s_s-1)} - \frac{A_1(s_s-1)(r_s-A_2)}{2r_s(r_s+s_s-1)} + \frac{2B_1B_2}{s_s} \tag{5.54}$$

and for (5.53) it is

$$\mathcal{F}^{(1)}_{W_3} = \frac{A_1A_2}{2r_s} + \frac{2B_1B_2}{s_s} \tag{5.55}$$

Combining the first two terms in (5.54) we will get (5.55). The details of the second order calculations can be found in [6].

Using Mathematica code we have checked up to the 8th order in z, that (5.36) agrees with (5.49).

Note that the expression (5.36), besides physical poles at $r_s \in \mathbb{Z}_{\leq 0}$, $s_s \in \mathbb{Z}_{\leq 0}$, $r_s + s_s - 1 \in \mathbb{Z}_{\leq 0}$, has apparent poles at positive integer values of s_s. In fact explicit calculations ensure that these apparent poles get canceled in final expressions.

Chapter 6

The light asymptotic limit of conformal blocks in N = 1 super Liouville field theory

6.1 The partition functions of $\mathcal{N} = 2$ SYM on R^4/Z_2

Let us consider $\mathcal{N} = 2$ SYM theory with a $U(2)$ gauge group on the space R^4/Z_2. The instanton part of the partition function for this theory can be represented as (see [136, 137])

$$Z^{(q_1,q_2)}_{(u_1,u_2),(v_1,v_2)}(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)}|q) = \sum_{\{\vec{Y}^{\vec{q}}\}} F_{\vec{Y}\,(u_1,u_2),(v_1,v_2)}^{\;(q_1,q_2)}\left(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)}\right) q^{\frac{|\vec{Y}|}{2}} \,. \tag{6.1}$$

The sum goes over the pairs of Young diagrams $\vec{Y}^{\vec{q}} = (Y_1^{q_1}, Y_2^{q_2})$ colored in chess like order. To each diagram one ascribes a $\mathbb{Z}_2$ charge q_i, $i = 1, 2$ which indicates the color of the corner and takes values 0 or 1 (white or black correspondingly). $|\vec{Y}|$ is the total number of boxes in Y_1 and Y_2, and q is the instanton counting parameter. Let us clarify our conventions on gauge theory parameters $a_i^{(0,1,2)}$, $i = 1, 2$. The parameters $a_i^{(1)}$ are expectation values of the scalar field in vector multiplet. Without loss of generality we will assume that the "center of mass" of these

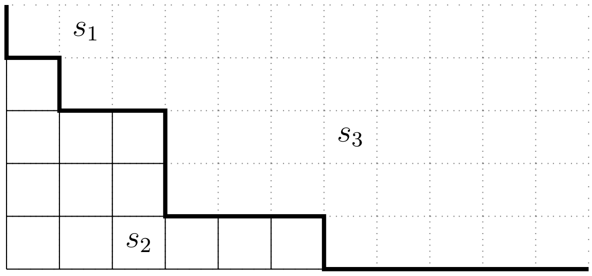

Figure 6.1: Arm and leg length with respect to the Young diagram whose borders are outlined by dark black: $A(s_1) = -2$, $L(s_1) = -2$, $A(s_2) = 2$, $L(s_2) = 3$, $A(s_3) = -3$, $L(s_3) = -4$.

expectation values is zero

$$\bar{a}^{(1)} = \frac{1}{2}\left(a_1^{(1)} + a_2^{(1)}\right) = 0\,, \tag{6.2}$$

since a nonzero center of mass can be absorbed by shifting hypermultiplet masses. Furthermore $a_i^{(0)}$ ($a_i^{(2)}$) are the masses of fundamental (anti-fundamental) hypers.

The expansion coefficient of the instanton partition function (6.1) is given by

$$F_{\vec{Y}\ (u_1,u_2),(v_1,v_2)}^{\ (q_1,q_2)}\left(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)}\right) = \tag{6.3}$$
$$\prod_{i=1}^{2}\prod_{j=1}^{2} \frac{Z_{bf}(u_i, a_i^{(0)}, \varnothing \mid q_j, a_j^{(1)}, Y_j) Z_{bf}(q_i, a_i^{(1)}, Y_i \mid v_j, a_j^{(2)}, \varnothing)}{Z_{bf}(q_i, a_i^{(1)}, Y_i \mid q_j, a_j^{(1)}, Y_j)}\,,$$

where

$$Z_{bf}(x, a, \lambda \mid y, b, \mu) = \tag{6.4}$$
$$\prod_{s\in\lambda^*}\left(a - b - \epsilon_1 L_\mu(s) + \epsilon_2(1 + A_\lambda(s))\right) \prod_{s\in\mu^*}\left(a - b + \epsilon_1(1 + L_\lambda(s)) - \epsilon_2 A_\mu(s)\right).$$

Here ϵ_1 and ϵ_2 are the Ω-background parameters. We will use the notation $\epsilon = \epsilon_1 + \epsilon_2$. $A_\lambda(s)$ ($L_\lambda(s)$) is the arm-length (leg-length) of the square s towards the Young diagram λ, defined as oriented vertical (horizontal) distance of the square s to outer boundary of the Young tableau λ (see figure 6.1). λ^*, μ^* are subsets of boxes λ and μ respectively such that, a box of λ (μ)

belongs to λ^* (μ^*) if and only if the replacement

$$\epsilon_1,\, \epsilon_2 \to 1;\;\; a \to x;\;\; b \to y \quad (i = 1, 2) \tag{6.5}$$

in the first (second) multiplier of (6.4) results in $0\,(mod\,2)$ (remind that u_i and v_i $(i = 1, 2)$ take values 0 or 1). For more details see [7].

According to the duality between $\mathcal{N} = 2$ SYM on R^4/Z_2 and $\mathcal{N} = 1$ SLFT these partition functions are directly related to four point conformal blocks in $\mathcal{N} = 1$ SLFT. Before describing this relation let us briefly recall few facts about $\mathcal{N} = 1$ SLFT itself.

6.2 More known facts on $\mathcal{N} = 1$ SLFT and its light asymptotic limit

In $N = 1$ super-Liouville field theory there are many kinds of primary fields let me list them in slightly more details then in 4.1

NS primary fields $\Phi_\alpha(z, \bar{z})$ in this theory, $\Phi_\alpha(z, \bar{z}) = e^{\alpha\varphi(z,\bar{z})}$, have conformal dimensions

$$\Delta_\alpha^{NS} = \frac{1}{2}\alpha(Q - \alpha)\,. \tag{6.6}$$

Introduce also the field that is the highest component of the NS superfield build from Φ_α

$$\Phi_{\tilde{\alpha}}(z, \bar{z}) = G_{-1/2}\bar{G}_{-1/2}\Phi_\alpha(z, \bar{z})\,, \tag{6.7}$$

with dimension

$$\tilde{\Delta}_\alpha^{NS} = \Delta_\alpha^{NS} + 1/2\,, \tag{6.8}$$

and as well as the Ramond primary fields defined as

$$R_{\alpha}^{\pm}(z,\bar{z}) = \sigma^{\pm}(z,\bar{z})e^{\alpha\varphi(z,\bar{z})} \tag{6.9}$$

where $\sigma^{\pm}$ is the spin field with dimension 1/16. Thus the dimension of a Ramond operator is

$$\Delta_{\alpha}^{R} = \frac{1}{16} + \frac{1}{2}\alpha(Q-\alpha)\,. \tag{6.10}$$

6.3 $\mathcal{N} = 1$ Super Liouville conformal blocks and their relation to the $\mathcal{N} = 2$ SYM on R^4/Z_2

Let us schematically denote by $\langle\Psi_1(\infty)\Psi_2(1)\Psi_3(q)\Psi_4(0)\rangle_{\Delta^{\Psi}}$ conformal block of Ψ_i, $i = 1\ldots 4$, fields with intermediary field Ψ of conformal weight Δ^{Ψ}.

Four point blocks where all four fields are bosonic primaries Φ_i with conformal weights Δ_{α_i} are connected with the Z_{inst} partition function in the following way (see [134])

$$^{\diamond}Z^{(0,0)}_{(0,0),(0,0)} = q^{\Delta_1^{NS}+\Delta_2^{NS}-\Delta^{NS}}(1-q)^{U}\langle\Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0)\rangle_{\Delta^{NS}} \tag{6.11}$$

and for $\tilde{\Delta} = \Delta + \frac{1}{2}$

$$^{\blacklozenge}Z^{(1,1)}_{(0,0),(0,0)} = \frac{q^{\Delta_1^{NS}+\Delta_2^{NS}-\tilde{\Delta}^{NS}}}{2}(1-q)^{U}\langle\Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0)\rangle_{\tilde{\Delta}^{NS}}\,. \tag{6.12}$$

The index $^{\diamond}$ shows that the number of black and white boxes (the number of boxes in both diagrams together) are equal and the index $^{\blacklozenge}$ show the number differ by one. In the expressions (6.11) and (6.12) U is given by

$$U = \alpha_2\left(Q-\alpha_3\right)\,. \tag{6.13}$$

We will see that in the light asymptotic limit U is just one. So in this limit the corresponding partiton function gives the four point conformal block for bosonic fields.

Let us look at the $\langle R\Phi\Phi R\rangle$ type conformal block. According to [133] this conformal blocks are connected to the instanton partition function in the following way

$$^{\diamond}Z^{(0,1)}_{(0,0),(0,0)} = q^{\Delta^R_3+\Delta^{NS}_4-\Delta^R}(1-q)^{(U-\frac{3}{8}+\Delta_1-\Delta_2-\Delta_3+\Delta_4)}\langle R^+_2(\infty)\Phi_1(1)\Phi_4(q)R^+_3(0)\rangle_{\Delta^R}. \quad (6.14)$$

Now let us look at the $\langle RRRR\rangle$ conformal blocks [133]. For the partition functions with equal numbers of black and white cells

$$^{\diamond}Z^{(0,0)}_{(1,0),(1,0)}(q) = (1-q)^U\left(G_{sl(2)}(q)H_-(q)+\tilde{G}_{sl(2)}(q)\tilde{H}_-(q)\right), \quad (6.15)$$

$$^{\diamond}Z^{(0,0)}_{(0,1),(0,1)}(q) = (1-q)^U\left(G_{sl(2)}(q)H_+(q)+\tilde{G}_{sl(2)}(q)\tilde{H}_+(q)\right), \quad (6.16)$$

$$^{\diamond}Z^{(0,0)}_{(1,0),(0,1)}(q) = (1-q)^U\left(G_{sl(2)}(q)F_-(q)+\tilde{G}_{sl(2)}(q)\tilde{F}_-(-q)\right), \quad (6.17)$$

$$^{\diamond}Z^{(0,0)}_{(0,1),(1,0)}(q) = (1-q)^U\left(G_{sl(2)}(q)F_+(q)+\tilde{G}_{sl(2)}(q)\tilde{F}_+(-q)\right). \quad (6.18)$$

For the partition functions whose numbers of black and white boxes differ by one

$$^{\blacklozenge}Z^{(1,1)}_{(1,0),(1,0)}(q) = (1-q)^U\left(\tilde{G}_{sl(2)}(q)H_+(q)+G_{sl(2)}(q)\tilde{H}_+(q)\right), \quad (6.19)$$

$$^{\blacklozenge}Z^{(1,1)}_{(0,1),(0,1)}(q) = (1-q)^U\left(\tilde{G}_{sl(2)}(q)H_-(q)+G_{sl(2)}(q)\tilde{H}_-(q)\right), \quad (6.20)$$

$$^{\blacklozenge}Z^{(1,1)}_{(1,0),(0,1)}(q) = (1-q)^U\left(\tilde{G}_{sl(2)}(q)F_+(q)+G_{sl(2)}(q)\tilde{F}_+(-q)\right), \quad (6.21)$$

$$^{\blacklozenge}Z^{(1,1)}_{(0,1),(1,0)}(q) = (1-q)^U\left(\tilde{G}_{sl(2)}(q)F_-(q)+G_{sl(2)}(q)\tilde{F}_-(-q)\right). \quad (6.22)$$

Here $H_\pm$, $F_\pm$, $\tilde{H}_\pm$ and $\tilde{F}_\pm$ are related to the conformal blocks containing four Ramond fields, for their definition see 6.6. $G(q)$ and $\tilde{G}(q)$ are certain conformal blocks of the $\hat{su(2)}_2$ WZW model, which are given by

$$G(q) = (1-q)^{-\frac{3}{8}}\sqrt{\tfrac{1}{2}\left(1+\sqrt{1-q}\right)}, \quad (6.23)$$

$$\tilde{G}(q) = (1-q)^{-\frac{3}{8}}\sqrt{\tfrac{1}{2}\left(1-\sqrt{1-q}\right)}. \quad (6.24)$$

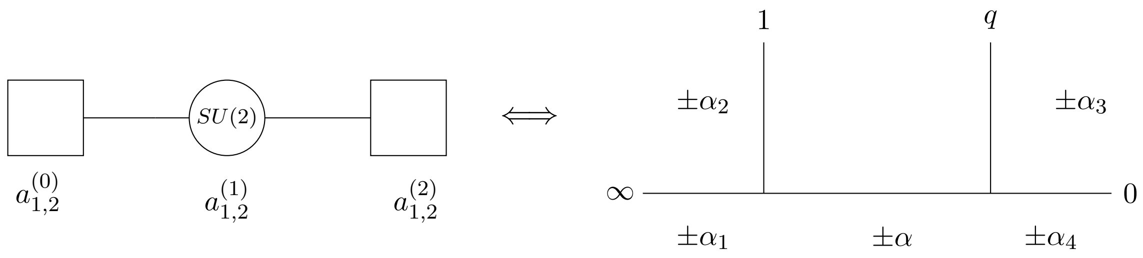

Figure 6.2: On the left: the quiver diagram for the conformal $SU(2)$ gauge theory. On the right: the diagram of the conformal block for the dual $\mathcal{N}=1$ SLFT .

Below is given the map that connects the gauge parameters of the instanton partition functions for $\mathcal{N}=2$ SYM on R^4/Z_2 to the primary fields in the $\mathcal{N}=1$ SLFT conformal blocks.

6.3.1 The map relating partition functions to conformal blocks

First of all, the instanton counting parameter q gets identified with the cross ratio of insertion points, as already anticipated in formulas (6.15)-(6.22), for CFT block. The Liouville parameter b is related to the Ω-background parameters via

$$b=\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,. \tag{6.25}$$

The map between the gauge parameters (6.1) and conformal block parameters can be established from the following rules (see Fig.6.2). First define the rescaled gauge parameters

$$A_i^{(0)}=\frac{a_i^{(0)}}{\sqrt{\epsilon_1\epsilon_2}}\,;\qquad A_i^{(1)}=\frac{a_i^{(1)}}{\sqrt{\epsilon_1\epsilon_2}}\,;\qquad A_i^{(2)}=\frac{a_i^{(2)}}{\sqrt{\epsilon_1\epsilon_2}}\,, \tag{6.26}$$

where $i=1,2$.

Then

- The differences between the "centers of masses" of the successive rescaled gauge parameters (6.26) give the charges of the "vertical" entries of the conformal block:

$$\bar{A}^{(1)} - \bar{A}^{(0)} = \alpha_2\,; \qquad \bar{A}^{(2)} - \bar{A}^{(1)} = \alpha_3\,. \tag{6.27}$$

- The rescaled gauge parameters with the subtracted centers of masses give the momenta of the "horizontal" entries of the conformal block:

$$\begin{aligned} A_i^{(0)} - \bar{A}^{(0)} &= (-)^{i+1}\left(\alpha_1 - \frac{Q}{2}\right)\,; \\ A_i^{(1)} - \bar{A}^{(1)} &= (-)^{i+1}\left(\alpha - \frac{Q}{2}\right)\,; \\ A_i^{(2)} - \bar{A}^{(2)} &= (-)^{i+1}\left(\alpha_4 - \frac{Q}{2}\right)\,. \end{aligned} \tag{6.28}$$

Using (6.2) and (6.26)-(6.28) we obtain the relation between the gauge and conformal parameters:

$$\begin{aligned} \frac{a_i^{(0)}}{\sqrt{\epsilon_1\epsilon_2}} &= (-)^{i+1}\left(\alpha_1 - \frac{Q}{2}\right) - \alpha_2\,; \\ \frac{a_i^{(1)}}{\sqrt{\epsilon_1\epsilon_2}} &= (-)^{i+1}\left(\alpha - \frac{Q}{2}\right)\,; \\ \frac{a_i^{(2)}}{\sqrt{\epsilon_1\epsilon_2}} &= (-)^{i+1}\left(\alpha_4 - \frac{Q}{2}\right) + \alpha_3\,. \end{aligned} \tag{6.29}$$

6.3.2 Light asymptotic limit of the gauge parameters

In this paper we are interested in so called "light" asymptotic limit i.e. the central charge is sent to infinity (i.e. $b \to 0$) while keeping the dimensions finite. It follows from (6.6) and (6.10) that to reach this limit one can simply put

$$\alpha = b\eta; \qquad \alpha_l = b\eta_l; \quad \text{where} \quad l = 1;\ 2;\ 4, \tag{6.30}$$

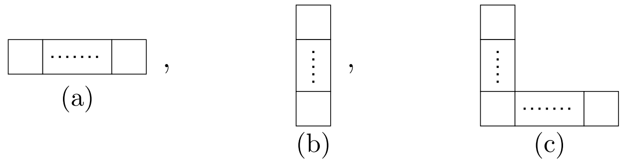

Figure 6.3: the possible nonempty Young diagrams

by keeping all the parameters η finite. If we exchange α with $Q-\alpha$ the conformal dimension remains the same (see (6.6) and (6.10)), so for α_3 we can take as its light asymptotic limit

$$Q - \alpha_3 = b\eta_3 \tag{6.31}$$

By taking the limit in this way we get rid of the $U(1)$ factor defined in (6.13). Using (6.30), (6.31) we can rewrite the AGT map (6.29) as

$$a_i^{(0)} = (-)^{i+1}\left(\epsilon_1\eta_1 - \frac{\epsilon}{2}\right) - \epsilon_1\eta_2 \tag{6.32}$$

$$a_i^{(1)} = (-)^{i+1}\left(\epsilon_1\eta - \frac{\epsilon}{2}\right) ; \tag{6.33}$$

$$a_i^{(2)} = (-)^{i+1}\left(\epsilon_1\eta_4 - \frac{\epsilon}{2}\right) + \epsilon - \epsilon_1\eta_3 \, . \tag{6.34}$$

6.4 Partition function in the light asymptotic limit

It is shown in [7] that for the light asymptotic limit only a restricted set of Young diagrams contributes to the instanton partition function. This set varies depending on the charges and the differences of black and white cells of the related Young diagrams. Below are given all pairs of Y_1 and Y_2 for which the coefficient of the instanton expansion (6.1) is non zero in the light limit. In order to compute these coefficients for a given pair of diagrams Y_1 and Y_2 one makes use of (6.3), (6.4), (6.32)-(6.34) and then goes to the light limit $\epsilon_1 \to 0$. The results are given below (detailed calculation for some of the coefficients can be found in [7]).

6.4.1 Partition functions corresponding to conformal blocks with four Neveu-Schwarz fields.

The expansion coefficient ${}^{\Diamond}F^{(0,0)}_{(0,0),(0,0)}$ does not vanish in the light asymptotic limit if Y_2 is a empty Young diagram and Y_1 (see figure 6.3(a)) has only one row with $2k$ boxes, where k can be zero or any positive integer. It is equal to

$$
{}^{\Diamond}_{L}F^{(0,0)}_{(0,0),(0,0)} = \frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3\right)\right)_k \left(\frac{1}{2}\left(\eta-\eta_1+\eta_2\right)\right)_k}{k!\left(\eta\right)_k} . \tag{6.35}
$$

Inserting (6.35) in (6.1), we derive

$$
{}^{\Diamond}_{L}Z^{(0,0)}_{(0,0),(0,0)}(q) = {}_2F_1\left(A, B; \eta; q\right) . \tag{6.36}
$$

Here A and B are

$$
A = \frac{1}{2}\left(\eta-\eta_1+\eta_2\right) \text{and} B = \frac{1}{2}\left(\eta-\eta_4+\eta_3\right) , \tag{6.37}
$$

and ${}_2F_1(a,b;c;x)$ is the hypergeometric function. It has the series expansion

$$
{}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} x^k \, , \text{where} (u)_k = u(u+1)\ldots(u+k-1) . \tag{6.38}
$$

In the case of ${}^{\blacklozenge}F^{(1,1)}_{(0,0),(0,0)}$ for some set of pairs Y_1, Y_2 one gets large coefficients of order $\frac{1}{\epsilon_1}$. Thus one should take into account these pairs and neglect those pairs whose contributions are of order $O(1)$ or bigger. An can show that Y_2 should be an empty and Y_1 must have a single row with $2k+1$ boxes (see figure 6.3(a)).Their contribution is

$$
{}^{\blacklozenge}_{L}F^{(1,1)}_{(0,0),(0,0)} = \frac{1}{\epsilon_1\epsilon_2} \frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3+1\right)\right)_k \left(\frac{1}{2}\left(\eta-\eta_1+\eta_2+1\right)\right)_k}{2\,k!\left(\eta\right)_{k+1}} . \tag{6.39}
$$

After inserting it in (6.1), we will get

$$
{}_{L}^{\blacklozenge}Z^{(1,1)}_{(0,0),(0,0)}(q) = \frac{1}{\epsilon_1\epsilon_2}\frac{\sqrt{q}}{2\eta}\,{}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;q\right). \tag{6.40}
$$

6.4.2 Partition function corresponding to the conformal block with two Neveu-Schwarz and two Ramond fields.

The coefficients of ${}^{\lozenge}Z^{(1,0)}_{(0,0),(0,0)}$ do not vanish in the light limit if Y_2 is empty and Y_1 (see figure 6.3(a)) is a diagram with only one row with $2k$ boxes. Their contributions are

$$
{}_{L}^{\lozenge}F^{(1,0)}_{(0,0),(0,0)} = \frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3+1\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2+1\right)\right)_k}{k!\left(\eta+\frac{1}{2}\right)_k}. \tag{6.41}
$$

The corresponding partition function is

$$
{}_{L}^{\lozenge}Z^{(1,0)}_{(0,0),(0,0)}(q) = {}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+\frac{1}{2};q\right). \tag{6.42}
$$

The case of ${}^{\lozenge}Z^{(0,1)}_{(0,0),(0,0)}$ is more subtle. Its coefficient do not vanish if Y_1 (see figure 6.3(a)) is a one row diagram with $2k$ boxes and Y_2 (see figure 6.3(b)) is a one column diagram with $2m$ boxes. Here one should consider the cases $m=0$ ans $m\neq 0$ separately:

- when $m=0$

$$
{}_{L}^{\lozenge}F^{(0,1)}_{(0,0),(0,0)} = \frac{\left(\frac{1}{2}\left(\eta-\eta^{(4)}+\eta^{(3)}\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta^{(1)}+\eta^{(2)}\right)\right)_k}{k!\left(\eta+\frac{1}{2}\right)_k}; \tag{6.43}
$$

- when $m\neq 0$

$$
{}_{L}^{\lozenge}F^{(0,1)}_{(0,0),(0,0)} = \frac{1}{2m+1}\frac{\left(\frac{1}{2}\left(\eta-\eta^{(4)}+\eta^{(3)}\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta^{(1)}+\eta^{(2)}\right)\right)_k}{k!\left(\eta-\frac{1}{2}\right)_k}. \tag{6.44}
$$

The corresponding instanton partition function is

$$ {}^{\Diamond}_{L}Z^{(0,1)}_{(0,0),(0,0)}(q) = {}_2F_1\left(A,B;\eta+\frac{1}{2};q\right) + \frac{\tanh^{-1}\left(\sqrt{q}\right)}{\sqrt{q}}\,{}_2F_1\left(A,B;\eta-\frac{1}{2};q\right) . \tag{6.45} $$

6.4.3 Partition functions corresponding to conformal blokes with four Ramond fields.

${}^{\Diamond}F^{(0,0)}_{(0,1),(0,1)}$ differs from zero in the light asymptotic limit if Y_2 (see figure 6.3(b)) is a single column diagram with $2m$ boxes, and Y_1 (see figure 6.3(a)) a single row diagram with $2k$ boxes, where m and k can be zero or any positive integer. Their contribution is

$$ {}^{\Diamond}_{L}F^{(0,0)}_{(0,1),(0,1)} = \left(\frac{(1/2)_m}{m!}\right)^2 \frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2\right)\right)_k}{k!\,(\eta)_k} . \tag{6.46} $$

Its instanton partition function is

$$ {}^{\Diamond}_{L}Z^{(0,0)}_{(0,1),(0,1)}(q) = \frac{2}{\pi}K(q)\,{}_2F_1\left(A,B;\eta;q\right) . \tag{6.47} $$

$K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind correspondingly. They can be expressed in terms of the Gauss hypergeometric function, as

$$ K(x) = \frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)\text{and}E(x) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;x\right) \tag{6.48} $$

In the case of ${}^{\Diamond}F^{(0,0)}_{(1,0),(1,0)}$ for pairs of Young diagrams Y_2, Y_1, with Y_2 empty and Y_1 (see figure 6.3(c)) possessing one column with $2m$ boxes and other $2k$ columns with only one box, one gets large coefficients of order $\frac{1}{\epsilon_1}$ in the light limit. In total Y_1 consists of $2m+2k$ boxes. These pairs give the main contribution. These terms are

$$ {}^{\Diamond}_{L}F^{(0,0)}_{(1,0),(1,0)} = \frac{\epsilon_2}{\epsilon_1}\frac{\left(\frac{1}{2}\right)_m\left(-\frac{1}{2}\right)_m}{(m-1)!m!}\frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3+1\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2+1\right)\right)_k}{k!\eta\,(\eta+1)_k} . \tag{6.49} $$

Its partition function is given by

$$ {}^{\diamondsuit}_{L}Z^{(0,0)}_{(1,0),(1,0)}(q) = \frac{\epsilon_2}{\epsilon_1}\frac{(E(q)-K(q))}{\pi\eta}\,{}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;q\right)\,. \tag{6.50} $$

${}^{\diamondsuit}_{L}F^{(0,0)}_{(0,1),(1,0)}$ differs from zero if Y_2 is empty and Y_1 is a one row diagram (see figure 6.3(a)) with $2k$ boxes. Their contribution is

$$ {}^{\diamondsuit}_{L}F^{(0,0)}_{(0,1),(1,0)} = \frac{\left(\frac{1}{2}(\eta-\eta_4+\eta_3+1)\right)_k\left(\frac{1}{2}(\eta-\eta_1+\eta_2)\right)_k}{k!\,(\eta)_k}\,. \tag{6.51} $$

Its instanton partition function is given by

$$ {}^{\diamondsuit}_{L}Z^{(0,0)}_{(0,1),(1,0)}(q) = {}_2F_1\left(A,B+\frac{1}{2};\eta;q\right)\,. \tag{6.52} $$

${}^{\diamondsuit}_{L}F^{(0,0)}_{(1,0),(0,1)}$ is not zero if Y_2 is empty and Y_1 (see figure 6.3(a)) is a one row diagram with $2k$ boxes. Their contribution is

$$ {}^{\diamondsuit}_{L}F^{(0,0)}_{(1,0),(0,1)} = \frac{\left(\frac{1}{2}(\eta-\eta_1+\eta_2+1)\right)_k\left(\frac{1}{2}(\eta-\eta_4+\eta_3)\right)_k}{k!\,(\eta)_k}\,. \tag{6.53} $$

Its partition function is given by

$$ {}^{\diamondsuit}_{L}Z^{(0,0)}_{(1,0),(0,1)}(q) = {}_2F_1\left(A+\frac{1}{2},B;\eta;q\right)\,. \tag{6.54} $$

In the case of ${}^{\blacklozenge}F^{(1,1)}_{(0,1),(0,1)}$ for some set of pairs Y_1, Y_2 one gets large coefficients of order $\frac{1}{\epsilon_1}$ in the light limit. These coefficients will give the main contribution in the partition function. These terms are obtained when Y_2 is empty and Y_1 (see figure 6.3(c)) has one column with $2m+1$ boxes and $2k$ columns with only one box, the total number of boxes is equal to $2m+2k+1$. They are given by

$$ {}^{\blacklozenge}_{L}F^{(1,1)}_{(0,1),(0,1)} = \frac{\epsilon_2}{\epsilon_1}\left(\frac{\left(\frac{1}{2}\right)_m}{m!}\right)^2\frac{\left(\frac{1}{2}(\eta-\eta_4+\eta_3+1)\right)_k\left(\frac{1}{2}(\eta-\eta_1+\eta_2+1)\right)_k}{-2\eta k!\,(\eta+1)_k}\,. \tag{6.55} $$

For its partition function, we receive

$$
{}^{\blacklozenge}_{L}Z^{(1,1)}_{(0,1),(0,1)}(q) = -\frac{\epsilon_2}{\epsilon_1}\frac{\sqrt{q}}{\pi\eta}K(q)\,{}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;q\right)\,. \tag{6.56}
$$

${}^{\blacklozenge}_{L}F^{(1,1)}_{(1,0),(1,0)}$ differs from zero if Y_2 is a one column diagram (see figure 6.3(b)) with $2m+1$ boxes and Y_1 is a one row diagram (see figure 6.3(a)) with $2k$ boxes. Their contribution is

$$
{}^{\blacklozenge}_{L}F^{(1,1)}_{(1,0),(1,0)} = \frac{1}{(2+2m)(1+2m)}\left(\frac{\left(\frac{3}{2}\right)_m}{m!}\right)^2\frac{\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2\right)\right)_k}{k!\,(\eta)_k}\,. \tag{6.57}
$$

For the corresponding instanton partition function, we will get

$$
{}^{\blacklozenge}_{L}Z^{(1,1)}_{(1,0),(1,0)}(q) = -\frac{2(E(q)-K(q))}{\pi\sqrt{q}}\,{}_2F_1\left(A,B;\eta;q\right)\,. \tag{6.58}
$$

Both ${}^{\blacklozenge}_{L}F^{(1,1)}_{(1,0),(0,1)}$ and ${}^{\blacklozenge}_{L}F^{(1,1)}_{(0,1),(1,0)}$ do not vanish if Y_2 is empty and Y_1 (see figure 6.3(a)) is a one row diagram with $2k+1$ boxes. Their contributions are

$$
{}^{\blacklozenge}_{L}F^{(1,1)}_{(0,1),(1,0)} = \frac{\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2+1\right)\right)_k\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3\right)\right)_{k+1}}{k!\,(\eta)_k}\,, \tag{6.59}
$$

$$
{}^{\blacklozenge}_{L}F^{(1,1)}_{(1,0),(0,1)} = \frac{\left(\frac{1}{2}\left(\eta-\eta_1+\eta_2\right)\right)_{k+1}\left(\frac{1}{2}\left(\eta-\eta_4+\eta_3+1\right)\right)_k}{k!\,(\eta)_k}\,. \tag{6.60}
$$

Their partition functions are

$$
{}^{\blacklozenge}_{L}Z^{(1,1)}_{(0,1),(1,0)}(q) = \frac{B}{\eta}\sqrt{q}\,{}_2F_1\left(A+\frac{1}{2},B+1;\eta+1;q\right)\,. \tag{6.61}
$$

$$
{}^{\blacklozenge}_{L}Z^{(1,1)}_{(1,0),(0,1)}(q) = \frac{A}{\eta}\sqrt{q}\,{}_2F_1\left(A+1,B+\frac{1}{2};\eta+1;q\right)\,. \tag{6.62}
$$

6.5 Conformal blocks for $\mathcal{N}=1$ SLFT in the light asymptotic limit

Applying (6.36) and (6.40) to (6.11) and (6.12) we will get the conformal blocks with all four fields being NS in the light limit:

$$\langle \Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0)\rangle^L_{\Delta^{NS}} = q^{\frac{1}{2}(\eta-\eta^{(2)}-\eta^{(1)})}\,{}_2F_1\left(A,B;\eta;q\right)\,, \tag{6.63}$$

$$\langle \Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0)\rangle^L_{\tilde{\Delta}^{NS}} = \frac{q^{\frac{1}{2}(1+\eta-\eta^{(2)}-\eta^{(1)})}}{\eta}\,{}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;q\right)\,. \tag{6.64}$$

These results are in agreement with [138].

By applying (6.45) for (6.14) we get the conformal blocks with two R fields and two NS fields

$$\begin{aligned}\langle R_2^+(\infty)\Phi_1(1)\Phi_4(q)R_3^+(0)\rangle^L_{\Delta^R} &= q^{\frac{1}{2}(\eta-\eta^{(3)}-\eta^{(4)})}(1-q)^{-\frac{1}{2}(\eta^{(1)}-\eta^{(2)}-\eta^{(3)}+\eta^{(4)}-1)}\\ &\left({}_2F_1\left(A,B;\eta+\frac{1}{2};q\right)+\frac{\tanh^{-1}\left(\sqrt{q}\right)}{\sqrt{q}}\,{}_2F_1\left(A,B;\eta-\frac{1}{2};q\right)\right)\,,\end{aligned} \tag{6.65}$$

where the intermediate field is a Ramond field.

As it was already mentioned the conformal blocks with four R fields are expressed in terms of $H_\pm$, $\tilde{H}_\pm$, $F_\pm$, $\tilde{F}_\pm$. Their connection to the instanton partition is given in (6.15)-(6.22). Applying (6.47)-(6.62), we can derive them. Their expressions get slightly simplified when one takes $q=\sin^2(t)$ with $t\in\left(0\,,\frac{\pi}{2}\right)$.

$$H_-^L(\sin^2(t)) = \frac{\epsilon_2}{\epsilon_1}\frac{\cos\left(\frac{t}{2}\right)\left(E\left(\sin^2(t)\right)-\cos(t)K\left(\sin^2(t)\right)\right){}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;\sin^2(t)\right)}{\pi\eta\sqrt[4]{\cos(t)}}\,, \tag{6.66}$$

$$\tilde{H}_-^L(\sin^2(t)) = -\frac{\epsilon_2}{\epsilon_1}\frac{\sin(t)\left(\cos(t)K\left(\sin^2(t)\right)+E\left(\sin^2(t)\right)\right){}_2F_1\left(A+\frac{1}{2},B+\frac{1}{2};\eta+1;\sin^2(t)\right)}{\sqrt{2}\pi\eta\sqrt[4]{\cos(t)}\sqrt{\cos(t)+1}}\,, \tag{6.67}$$

$$H_+^L(\sin^2(t)) = \frac{\sec\left(\frac{t}{2}\right)\left(\cos(t)K\left(\sin^2(t)\right)+E\left(\sin^2(t)\right)\right){}_2F_1\left(A,B;\eta;\sin^2(t)\right)}{\pi\sqrt[4]{\cos(t)}}\,, \tag{6.68}$$

$$\tilde{H}_+^L(\sin^2(t)) = \frac{\csc\left(\frac{t}{2}\right)\left(\cos(t)K\left(\sin^2(t)\right)-E\left(\sin^2(t)\right)\right){}_2F_1\left(A,B;\eta;\sin^2(t)\right)}{\pi\sqrt[4]{\cos(t)}}\,, \tag{6.69}$$

$$F_+^L(\sin^2(t)) = \frac{\sec\left(\frac{t}{2}\right)\left(\eta(\cos(t)+1)\,{}_2F_1\left(A,B+\frac{1}{2};\eta;\sin^2(t)\right)-A\sin^2(t)\,{}_2F_1\left(A+1,B+\frac{1}{2};\eta+1;\sin^2(t)\right)\right)}{2\eta\sqrt[4]{\cos(t)}}\,, \tag{6.70}$$

$$F_-^L(\sin^2(t)) = \frac{\sec\left(\frac{t}{2}\right)\left(\eta(\cos(t)+1)\,{}_2F_1\left(A+\frac{1}{2},B;\eta;\sin^2(t)\right)-B\sin^2(t)\,{}_2F_1\left(A+\frac{1}{2},B+1;\eta+1;\sin^2(t)\right)\right)}{2\eta\sqrt[4]{\cos(t)}}\,, \tag{6.71}$$

$$\tilde{F}_+^L(-\sin^2(t)) = \frac{\sin(t)\left(A(\cos(t)+1)\,{}_2F_1\left(A+1,B+\frac{1}{2};\eta+1;\sin^2(t)\right)-\eta\,{}_2F_1\left(A,B+\frac{1}{2};\eta;\sin^2(t)\right)\right)}{\sqrt{2}\eta\sqrt[4]{\cos(t)}\sqrt{\cos(t)+1}}\,, \tag{6.72}$$

$$\tilde{F}_-^L(-\sin^2(t)) = \frac{\sin(t)\left(B(\cos(t)+1)\,{}_2F_1\left(A+\frac{1}{2},B+1;\eta+1;\sin^2(t)\right)-\eta\,{}_2F_1\left(A+\frac{1}{2},B;\eta;\sin^2(t)\right)\right)}{\sqrt{2}\eta\sqrt[4]{\cos(t)}\sqrt{\cos(t)+1}}\,. \tag{6.73}$$

6.6 Super Liouville conformal blocks of four R-fields

Here, following [133] we define the functions $H_\pm$, $F_\pm$, $\tilde{H}_\pm$ and $\tilde{F}_\pm$, which are used in the main text. The OPEs for two Ramond fields can be written as

$$R_1^\pm(z)R_2^\pm(0) = z^{\Delta-\Delta_1-\Delta_2}\sum_{N=0}^{\infty} z^N|N;\pm\pm\rangle\,, \tag{6.74}$$

$$R_1^\pm(z)R_2^\mp(0) = z^{\Delta-\Delta_1-\Delta_2}\sum_{N=0}^{\infty} z^N|N;\pm\mp\rangle\,. \tag{6.75}$$

In the NS sector at level zero there is only one state, namely the NS primary state of dimension Δ. Thus $|N;\pm\pm\rangle$ states are proportional to this NS state

$$|0;\pm\pm\rangle = \gamma_\pm|0\rangle\,. \tag{6.76}$$

By definition

$$|N;\pm\rangle = |N;++\rangle \pm |N;--\rangle \quad \text{if} N\in Z \tag{6.77}$$

$$|N;+-\rangle \mp i|N;-+\rangle \quad \text{if} N\in Z+1/2 \tag{6.78}$$

In this notations

$$|0;\pm\rangle = \Gamma_\pm|0\rangle \quad \text{where} \quad \Gamma_\pm = (\gamma_+ \pm \gamma_-)\,. \tag{6.79}$$

$H_\pm$, $F_\pm$, $\tilde{H}_\pm$ and $\tilde{F}_\pm$ are related to the conformal blocks with four Ramond fields in the following way (below q is the cross ratio of insertion points)

$$F_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\pm}} \sum_{N=0,1,\ldots} q^N \langle N;\pm|N;\pm\rangle\,; \quad H_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\mp}} \sum_{N=0,1,\ldots} q^N \langle N;\pm|N;\mp\rangle\,, \tag{6.80}$$

$$\tilde{F}_{\pm} = \frac{(-i)}{\Gamma_{\pm}\Gamma_{\pm}} \sum_{N=\frac{1}{2},\frac{3}{2},\ldots} q^N \langle N;\pm|N;\pm\rangle\,; \quad \tilde{H}_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\mp}} \sum_{N=\frac{1}{2},\frac{3}{2},\ldots} q^N \langle N;\pm|N;\mp\rangle\,, \tag{6.81}$$

where conformal blocks are divided by $\Gamma_{\pm}$ so that if one takes the normalization $\langle 0|0\rangle = 1$, then the expansion of $F_{\pm}$ starts as $1 + F_{\pm 1}q + \ldots$. For more details and explanation the reader should consult [133].

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