Basis of two-dimensional conformal field theory with defects and boundaries

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Content

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Lecture 1
Path integrals

Gauss integrals

Let us start recalling the formula

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \tag{1}$$

This formula is equivalent to equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dxdy = \frac{\pi}{\alpha} \tag{2}$$

which can be easily derived in the polar coordinates (r, θ):

$$\int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2} r dr d\theta == 2\pi \int_0^{\infty} e^{-\alpha r^2} r dr = \pi \int_0^{\infty} e^{-\alpha r^2} d(r)^2 = \frac{\pi}{\alpha} \tag{3}$$

Thus the relation (1) is proved.

Completing the square we can also prove:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \exp\left(\frac{b^2}{4a}+c\right)\left(\frac{\pi}{a}\right)^{1/2} \tag{4}$$

This can be generalized to higher dimensions:

$$\int \exp\left[-\left(\frac{1}{2}(x, Ax) + (b, x) + c\right)\right] dx = (2\pi)^{n/2} \exp\left[\frac{1}{2}(b, A^{-1}b) - c\right] (\det A)^{-1/2} \tag{5}$$

where A is $n \times n$ matrix, and x, b, c are n-dimensional vectors.

Another important integrals:

$$\int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = 0 \tag{6}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2} \tag{7}$$

Action

$$S = \int_{t_i}^{t_f} L(q, \dot{q}, t) \tag{8}$$

$$\delta S = \frac{\partial L}{\partial \dot{q}}|_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q \delta t \tag{9}$$

At the equation of motion the second term equals to 0. In the first term we take $\delta q(t_1) = 0$. Setting $\frac{\partial L}{\partial \dot{q}} = p$ we obtain

$$\delta S = p\delta q \tag{10}$$

From here we obtain:

$$\frac{\partial S}{\partial q} = p \tag{11}$$

$$\frac{dS}{dt} = L \tag{12}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\dot{q} = \frac{\partial S}{\partial t} + p\dot{q} \tag{13}$$

Comparing (12) and (13) we obtain

$$\frac{\partial S}{\partial t} = L - p\dot{q} \tag{14}$$

or finally

$$\frac{\partial S}{\partial t} = -H \tag{15}$$

Path integral

Let us start by the formula:

$$K(q_f, t_f; q_i, t_i) = \int \mathcal{D}q e^{iS/\hbar} \tag{16}$$

In the classical limit

$$K(q_f, t_f; q_i, t_i) \sim \exp\left[\frac{i}{\hbar} S_{\rm cl}(q_f, t_f; q_i, t_i)\right. \tag{17}$$

Change of the action caused by the variation of the last point position can be evaluated with help of the formulas (11) and (15) leading to

$$K(q_f, t_f; q_i, t_i) \sim \exp\left[\frac{i}{\hbar}(pq_f + Et_f)\right] \tag{18}$$

This implies de Broglie relations:

$$p = k\hbar \tag{19}$$

$$E = \omega\hbar \tag{20}$$

As a consequence of the relation

$$S(q_f, t_f; q_i, t_i) = S(q_f, t_f; q, t) + S(q, t; q_i, t_i) \tag{21}$$

the propagation amplitude (16) satisfies the relation:

$$K(q_f,t_f;q_i,t_i) = \int K(q_f,t_f;q,t)K(q,t;q_i,t_i)dq \tag{22}$$

The wave function satisfies the relation

$$\psi(q_2,t_2) = \int K(q_2,t_2;q_1,t_1)\psi(q_1,t_1)dq_1 \tag{23}$$

To evaluate the path integral, we must define the symbol $\int \mathcal{D}q$. We will use a brute-force definition, by discretization. Break up the time interval from t_i to t_f into many small pieces of duration δt. Approximate a path as a sequence of straight line, one in each slice. The action for this discretized path is

$$S = \int_{t_i}^{t_f} dt\left(\frac{m}{2}\dot{q}^2 - U(q)\right) \to \sum_k \left[\frac{m}{2}\frac{(q_{k+1}-q_k)^2}{\delta t} - \delta t U\left(\frac{q_{k+1}+q_k}{2}\right)\right] \tag{24}$$

We then define the path integral by

$$\int \mathcal{D}q = \frac{1}{C(\delta t)}\int\frac{dq_1}{C(\delta t)}\int\frac{dq_2}{C(\delta t)}\cdots\int\frac{dq_{N-1}}{C(\delta t)} = \frac{1}{C(\delta t)}\prod_k\int\frac{dq_k}{C(\delta t)} \tag{25}$$

To derive differential equation satisfied by the wave function, consider the equation (23 for the case when $t_2 - t_1 = \delta t$. We should have

$$\psi(q_2,t) = \int\frac{dq_1}{C(\delta t)}\exp\left[i\frac{m(q_2-q_1)^2}{2\delta t} - i\delta t U\left(\frac{q_2+q_1}{2}\right)\right]\psi(q_1,t-\delta t) \tag{26}$$

As we send $\delta t \to 0$ the rapid oscillation of the first term in the exponential constraints q_1 to be very close to q_2. We can therefore expand the above expression of $q_1 - q_2 \equiv \eta$:

$$\begin{aligned}\psi(q_2,t) &= \int\frac{dq_1}{C(\delta t)}\exp\left[i\frac{m(q_2-q_1)^2}{2\delta t}\right]\left[1 - i\delta t U(q_2) + \cdots\right] \\ &\times\left[1 + (q_1-q_2)\frac{\partial}{\partial q_2} + \frac{1}{2}(q_1-q_2)^2\frac{\partial^2}{\partial q_2^2} + \cdots\right]\psi(q_2,t-\delta t) \\ &= \int\frac{d\eta}{C(\delta t)}\exp\left[i\frac{m\eta^2}{2\delta t}\right]\left[1 - i\delta t U(q_2) + \cdots\right]\left[1 + \eta\frac{\partial}{\partial q_2} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial q_2^2} + \cdots\right]\psi(q_2,t-\delta t)\end{aligned} \tag{27}$$

$$\psi(q_2,t) = \left(\frac{1}{C(\delta t)}\sqrt{\frac{2\pi\delta t}{-im}}\right)\left[1 - i\delta t U(q_2) + \frac{i\delta t}{2m}\frac{\partial^2}{\partial q_2^2} + O(\delta t^2)\right]\psi(q_2,t-\delta t) \tag{28}$$

This expression makes no sense in the limit $\delta t \to 0$ unless the factor in parentheses equal 1. We can therefore identify the correct definition of $C(\delta t)$:

$$C(\delta t) = \sqrt{\frac{2\pi\delta t}{-im}} \tag{29}$$

Given this definition, we can compare terms of order δt

$$i\frac{\partial}{\partial t}\psi(q_2, t) = \Big[-\frac{1}{2m}\frac{\partial^2}{\partial q_2^2} + U(q_2)\Big]\psi(q_2, t) \tag{30}$$

This is the Schrödinger equation.

Using the equation (22) repeating the same steps we obtain that also the probability amplitude satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t_2}K(q_2, t_2, q_1, t_1) = \Big[-\frac{1}{2m}\frac{\partial^2}{\partial q_2^2} + U(q_2)\Big]K(q_2, t_2, q_1, t_1) \tag{31}$$

for $t_2 > t_1$.

The function $K(q_2, t_2, q_1, t_1)$ is set to 0 for $t_2 < t_1$.

Now consider the $t_2 \to t_1$. The function $K(q_2, t_2, q_1, t_1)$ in this limit is

$$\frac{1}{C(\delta t)}\exp\Big[i\frac{m(q_2 - q_1)^2}{2\delta t} + O(\delta t)\Big] \tag{32}$$

This is peaked exponential and it tends to $\delta(q_2 - q_1)$ as $\delta t \to 0$. Therefore it behaves like step function at 0 and being differentiated by t_2 gives rise to $\delta(t_2 - t_1)$ multiplied by $\delta(q_2 - q_1)$:

$$i\frac{\partial}{\partial t_2}K(q_2, t_2, q_1, t_1) = \Big[-\frac{1}{2m}\frac{\partial^2}{\partial q_2^2} + U(q_2)\Big]K(q_2, t_2, q_1, t_1) + i\delta(t_2 - t_1)\delta(q_2 - q_1) \tag{33}$$

Therefore $K(q_2, t_2, q_1, t_1)$ is Green function of the Schrödinger equation.

Evolution operator

Now compute the matrix element of the evolution operator of the one degree of freedom system

$$\langle q_f | U(t) | q_i \rangle \tag{34}$$

The Hamiltonian of the system is

$$H = \frac{\hat{p}^2}{2m} + U(\hat{q}) \tag{35}$$

The hat denotes the corresponding quantum operator. The evolution operator which takes a state $|\psi\rangle$ at time t_i to the time $t_f = t_i + t$ is

$$U(t) = e^{-iHt} \tag{36}$$

We calculate the matrix element of $U(\delta t)$ in the basis $\{|q\rangle\}$ of position eigenstates, where δt is an infinitesimal time interval. At the first order in δt one has:

$$\begin{aligned}\langle q'|e^{-i\left(\frac{\hat{p}^2}{2m}+U(\hat{q})\right)\delta t}|q\rangle &= \langle q'|e^{-i\left(\frac{\hat{p}^2\delta t}{2m}\right)}e^{-i(U(\hat{q}))\delta t}e^{O(\delta t)^2}|q\rangle \\ &= \int \frac{dp}{2\pi}\langle q'|e^{-i\left(\frac{\hat{p}^2\delta t}{2m}\right)}|p\rangle\langle p|e^{-i(U(\hat{q}))\delta t}|q\rangle \\ = \int \frac{dp}{2\pi}\exp\Big[&- i\delta t\left(\frac{p^2}{2m} - p\frac{(q'-q)}{\delta t} + U(q)\right)\Big] \\ = \sqrt{\frac{m}{2\pi i\delta t}}\exp&\left[i\delta t\left(\frac{1}{2}m\frac{(q'-q)^2}{\delta t^2} - U(q)\right)\right]\end{aligned} \tag{37}$$

At the first step we used that

$$e^{\epsilon(A+B)} = e^{\epsilon A}e^{\epsilon B}e^{O(\epsilon^2)} \tag{38}$$

In the second step the terms of order $(\delta t)^2$ have been neglected and inserted a completeness relation

$$\int \frac{dp}{2\pi}|p\rangle\langle p| = 1 \tag{39}$$

where $|p\rangle$ is an eigenstate of momentum, with $\langle x|p\rangle = e^{ipx}$. In the last step we performed a Gauss integration (4) which is strictly valid only when the time interval δt has a small negative imaginary part. This assumption will be implicit in what follows. The quantity in brackets on the last line of (37) is nothing but the infinitesimal action $S(q', t_i + \delta t; q, t_i)$ corresponding to the passage of the system from q to q' in a time δt. One may therefore write, to first order:

$$\langle q'|U(\delta t)|q\rangle = \sqrt{\frac{m}{2\pi i\delta t}}\exp iS(q', t_i + \delta t; q, t_i) \tag{40}$$

Now we consider $\langle q_f|U(t)|q_i\rangle$ which is probability amplitude for the system, initially at a well-defined position q_i, to evolve in a finite time t toward the position q_f. This amplitude is called propagator and may be obtained by dividing the interval of time t in N subintervals t/N and inserting completeness relations:

$$\begin{aligned}&\langle q_f|U(t)|q_i\rangle = \\ &\int \prod_{j=1}^{N-1} dq_j \langle q_f|U(t/N)|q_{N-1}\rangle\langle q_{N-1}|U(t/N)|q_{N-2}\rangle \cdots \langle q_1|U(t/N)|q_i\rangle\end{aligned} \tag{41}$$

Using (40) in the large N limit one may write:

$$\langle q_f|U(t)|q_i\rangle = \lim_{N\to\infty}\left(\frac{mN}{2\pi it}\right)^{N/2}\int\prod_{j=1}^{N-1}dq_j\exp iS[q] \tag{42}$$

where

$$S[q]=\sum_{j=0}^{N-1}S(q_{j+1},t_j+t/N;q_j,t_j) \tag{43}$$

is the action associated with the discrete trajectory $q_j, j=0,1\cdots N$ ($q_0=q_i$, $t_0=t_i$ and $q_N=q_f$, $t_N=t_f$). If we define the following functional integration measure

$$\mathcal{D}q = \lim_{N\to\infty}\prod_{j=1}^{N-1}\left(\frac{mN}{2\pi it}\right)^{1/2}dq_j \tag{44}$$

we may then write our fundamental result as follows:

$$\langle q_f|U(t)|q_i\rangle = \int\mathcal{D}q\exp iS(q_f,t_f;q_i,t_i) \tag{45}$$

where the action is

$$S(q_f,t_f;q_i,t_i)=\int_{t_i}^{t_f}dt\left(\frac{1}{2}m\dot{q}^2-U(q)\right) \tag{46}$$

for $q(t)$ with the boundary conditions $q(t_f)=q_f$ and $q(t_i)=q_i$.

Correlation function

This formalism may be extended to matrix elements of operators. Suppose, for instance, that we want to compute the matrix element of an operator $O(q)$ at an intermediate time t between t_i and t_f:

$$\langle q_f,t_f|O(t)|q_i,t_i\rangle = \langle q_f|e^{-iH(t_f-t)}Oe^{-iH(t-t_i)}|q_i\rangle \tag{47}$$
$$=\int dq'dq''\int\mathcal{D}qe^{iS(q_f,t_f;q'',t)}\langle q''|O|q'\rangle\int\mathcal{D}qe^{iS(q',t;q,t_i)}$$

Let us assume that O is diagonal in the q representation

$$\langle q''|O|q'\rangle = O(q')\delta(q''-q') \tag{48}$$

The above expression reduces symbolically to

$$\langle q_f,t_f|O(t)|q_i,t_i\rangle = \int\mathcal{D}qe^{iS(q_f,t_f;q_i,t_i)}O[q(t)] \tag{49}$$

This may be further generalized to a time-ordered product of operators:

$$O_1(t_1)O_2(t_2)\cdots \qquad t_1 \geq t_2 \geq \cdots \tag{50}$$

If they all diagonal in the q representation we have:

$$\langle q_f, t_f|O_1(t_1)O_2(t_2)\cdots|q_i, t_i\rangle = \int \mathcal{D}q e^{iS(q_f,t_f;q_i,t_i)} O_1[q(t_1)]O_2[q(t_2)]\cdots \tag{51}$$

The above expression can be generalized to field theory:

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\Phi(x_1)\cdots\Phi(x_n)\exp -S[\Phi] \tag{52}$$

where Z is the vacuum functional.

Partition function

The partition function can be expressed as trace of the density operator ρ

$$Z = \mathrm{Tr}\rho \tag{53}$$

where

$$\rho = \exp -\beta H \tag{54}$$

where $\beta = \frac{1}{T}$ is the inverse temperature. The resemblance between the density opeartor $e^{-\beta H}$ and the evolution operator e^{-itH} allows for the representation of the density operator as a functional integral. This introduces the lagrangian formalism into statistical mechanics. Explicitly consider the kernel of the density operator for a single degree of freedom

$$\rho(x_f, x_i) = \langle x_f|e^{-\beta H}|x_i\rangle \tag{55}$$

The path integral is adapted to this kernel by substituting $t \to -i\tau$ (the Wick rotation) where τ is a real variable going from 0 to β. The kernel of the density operator ρ becomes then

$$\rho(x_f, x_i) = \int_{x_i,0}^{x_f,\beta} \mathcal{D}x \exp -S[x] \tag{56}$$

The partition function may be expressed

$$Z = \int dx\rho(x,x) = \int \mathcal{D}x \exp -S[x] \tag{57}$$

This time the integration limits are no longer specified: all closed trajectories $x(0) = x(\beta)$ contribute.

Wick theorem

$$T[A(x_1)A(x_2)] =: A(x_1)A(x_2) : +\langle 0|TA(x_1)A(x_2)|0\rangle \tag{58}$$

$$\begin{aligned} T[A(x_1)\cdots A(x_n)] &=: A(x_1)\cdots A(x_n) : \\ +\sum_{k<l} : A(x_1)\cdots A(\hat{x_k})\cdots A(\hat{x_l})\cdots A(x_n) : \langle 0|TA(x_k)A(x_l)|0\rangle &+ \cdots \\ + \sum_{k_1<k_2<\cdots<k_{2p}} &: A(x_{k_1})\cdots A(\hat{x}_{k_{2p}})\cdots A(\hat{x_l})\cdots A(x_n) : \\ \times \sum_P \langle 0|TA(x_{P_1})A(x_{P_2})|0\rangle \cdots \langle 0|TA(x_{P_{2p-1}})A(x_{P_{2p}})|0\rangle &+ \cdots \end{aligned} \tag{59}$$

This may be further extended to an expression of the form

$$T : [A(x_1)\cdots A(x_k)] : \cdots : [A(x_l)\cdots A(x_n)] : \tag{60}$$

with the restriction that only contraction between distinct normal ordered products occur.

Lecture 2
Groups and Algebras Lie

Lie groups are groups of transformations $T(\theta)$ that are described by a finite set of real continuous parameters θ^a.

The group multiplication law then takes the form

$$T(\phi)T(\theta) = T\left(f(\phi,\theta)\right) \tag{61}$$

with $f^a(\phi,\theta)$ a function of the ϕ s and θ s. Taking $\theta^a = 0$ as the coordinates of the identity $T(0) = e$, we must have

$$f^a(\theta,0) = f^a(0,\theta) = \theta^a \tag{62}$$

The transformation of such continuous groups must be represented on the physical Hilbert space by unitary operators $U(T(\theta))$. These operators can be represented in at least a finite neighborhood of the identity by a power series:

$$U(T(\theta)) = 1 + i\theta^a X_a + \frac{1}{2}\theta^b\theta^c X_{bc} + \cdots \tag{63}$$

where

$$X_a = -i\frac{\partial U(T(\theta))}{\partial\theta^a}\bigg|_{\theta=0} \tag{64}$$

Consider the product of the two elements $U(T(\theta))$ and $U(T(\phi))$

$$U(T(\phi))U(T(\theta)) = U(T\left(f(\phi,\theta)\right)) \tag{65}$$

According to (62) the expansion of $f(\phi,\theta)$ to second order must take the form:

$$f^a(\phi,\theta) = \phi^a + \theta^a + f^a_{bc}\phi^b\theta^c + \cdots \tag{66}$$

with real coefficients f^a_{bc}. The presence of any terms of order ϕ^2 and θ^2 would violate (62). Then (65) reads:

$$[1 + i\phi^a X_a + \frac{1}{2}\phi^b\phi^c X_{bc} + \cdots] \times [1 + i\theta^a X_a + \frac{1}{2}\theta^b\theta^c X_{bc} + \cdots] = \tag{67}$$
$$1 + i(\phi^a + \theta^a + f^a_{bc}\phi^b\theta^c + \cdots)X_a + \frac{1}{2}(\phi^b + \theta^b + \cdots)(\phi^c + \theta^c + \cdots)X_{bc}$$

The terms of order 1, ϕ, θ, ϕ^2, θ^2 automatically match on both sides of Eq. (67),but from the $\phi\theta$ terms we obtain a non-trivial condition:

$$X_{bc} = -X_b X_c - i f^a_{bc} X_a \tag{68}$$

This shows that if we are given the structure of the group, i.e. the functions $f^a(\phi, \theta)$, and hence its quadratic coefficients f^a_{bc}, we can calculate the second order terms in $U(T(\theta))$ from the generators appearing in the first order terms. However, there is a consistency condition: the operator X_{bc} must be symmetric in b and c (because it is the second derivative of $U(T(\theta))$ with respect to θ^b and θ^c) so Eq. (68) requires that

$$[X_b, X_c] = iC^a_{bc} X_a \tag{69}$$

where C^a_{bc} are a set of real constants known as structure constants

$$C^a_{bc} = -f^a_{bc} + f^a_{cb} \tag{70}$$

Such a set of commutation relations is known as a Lie algebra. For any integer N

$$U(T(\theta)) = \left[U\left(T\left(\frac{\theta}{N}\right)\right)\right]^N \tag{71}$$

Letting $N \to \infty$ and keeping only the first-order terms in $U(T(\theta/N))$ we have then

$$U(T(\theta)) = \lim_{N\to\infty}\left[1 + \frac{i}{N}\theta^a X_a\right]^N \tag{72}$$

and hence

$$U(T(\theta)) = \exp(i\theta^a X_a) \tag{73}$$

Adjoint representation

Jacoby identity

$$[X_n, [X_b, X_c]] + [X_c, [X_n, X_b]] + [X_b, [X_c, X_n]] = 0 \tag{74}$$

From here we have:

$$C^l_{bc} C^m_{nl} + C^l_{nb} C^m_{cl} + C^l_{cn} C^m_{bl} = 0 \tag{75}$$

Representation is given by matrices $R(X)$ satisfying

$$R(X_b)_{ml} R(X_c)_{ln} - R(X_c)_{ml} R(X_b)_{ln} = iC^l_{bc} R(X_l)_{mn} \tag{76}$$

Let us take

$$R(X_b)_{mn} = iC^m_{bn} \tag{77}$$

Substituting (77) in (76) we have:

$$C^m_{bl} C^l_{cn} - C^m_{cl} C^l_{bn} = C^l_{bc} C^m_{ln} \tag{78}$$

what coincide with the Jacoby identities (75). We proved that (77) gives as the representation of the Lie algebra. This representation is called adjoint representation.

Global symmetries and Noether current

Consider the action

$$S = \int \mathcal{L}(\Phi, \partial_\mu \Phi) d^N x \tag{79}$$

Consider the transformation

$$\delta_\omega \Phi = -i\omega_a G_a \Phi \tag{80}$$

Here ω_a are infinitesimal parameter of transformation, G_a are generators of transformations, forming the algebra Lie. The transformations are global symmetry of the system if under (80) with the constant ω_a the action is invariant: $\delta S = 0$. If we now consider the same transformations but with ω an arbitrary function of position in spacetime

$$\delta_\omega \Phi = -i\omega_a(x) G_a \Phi \tag{81}$$

then in general, the variation of the action will not vanish, but it will have to be of the form:

$$\delta S = \int J^\mu_a \frac{\partial \omega_a}{\partial x^\mu} d^N x \tag{82}$$

in order that it should vanish when $\omega_a(x)$ is constant. If we now take the fields in $S(\Phi)$ to satisfy the field equation the S is stationary with respect to arbitrary field variations including variation of the form (81) so in this case (82) should vanish. Integrating by parts we see that J^μ_a must satisfy a conservation law:

$$\partial_\mu J^\mu_a = 0 \tag{83}$$

It follows immediately that

$$\frac{dQ_a}{dt} = 0 \tag{84}$$

where

$$Q_a = \int d^{N-1} x J^0_a \tag{85}$$

There is one such conserved current J^μ_a and one constant of motion Q_a for each independent infinitesimal symmetry transformation. This represent a general feature of the canonical formalism, often referred to as Noethers theorem: symmetries imply conservation laws.

Ward identity

Denoting by X the collection of the fields $\Phi(X_1)\cdots\Phi(X_n)$ one can write according to (52)

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\Phi X \exp(-S[\Phi]) \tag{86}$$

Changing the integration variables according the (81), namely

$$\mathcal{F}(\Phi) = \Phi(x) - i\omega_a(x) G_a \Phi(x) \tag{87}$$

will not change the path integral

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\mathcal{F}(\Phi)(X+\delta X)\exp - \Big[S[\Phi] + \int d^N x J^\mu_a \partial_\mu \omega_a\Big] \tag{88}$$

and hence assuming the invariance of the measure one has in the first order

$$\int \mathcal{D}\Phi \delta X \exp(-S[\Phi]) + \int \mathcal{D}\Phi X \exp(-S[\Phi])\Big(\int dx J^\mu_a \partial_\mu \omega_a\Big) = 0 \tag{89}$$

or

$$\langle \delta X\rangle = -\int dx \langle J^\mu_a(x) X\rangle \partial_\mu \omega_a \tag{90}$$

The variation δX is explicitly given by

$$\delta X = -i\sum_{i=1}^{n}(\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n))\omega_a(x_i) \tag{91}$$

$$= -i\int dx \omega_a(x)\sum_{i=1}^{n}(\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n))\delta(x-x_i)$$

Since (90) holds for any infinitesimal function $\omega_a(x)$ we may write the following local relation:

$$\frac{\partial}{\partial x^\mu}\langle J^\mu_a(x)\Phi(x_1)\cdots\Phi(x_n)\rangle \tag{92}$$

$$= -i\sum_{i=1}^{n}\langle \Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\rangle\delta(x-x_i)$$

The Ward identity allows us to identify the conserved charge Q_a (85) as the generator of the symmetry transformation in the Hilbert space of quantum states. Let $Y = \Phi(x_2)\cdots\Phi(x_n)$ and suppose that the time $t \equiv x_1^0$ is different from all the times in Y. We integrate the Ward identity (92) in a very thin box bounded by $t_- < t$ by $t_+ > t$ and by spatial infinity, which excludes all the other points $x_2 \cdots x_n$. Th integral of l.h.s of (92) is converted into a surface integral and yields:

$$\langle Q_a(t_+)\Phi(x_1)Y\rangle - \langle Q_a(t_-)\Phi(x_1)Y\rangle = -i\langle G_a\Phi(x_1)Y\rangle \tag{93}$$

Recalling that a correlation function is the vacuum expectation value of a time-ordered product in the operator formalism, and assuming, for the sake of argument, that all other times x_i^0 are less than t we write in the limit $t_- \to t_+$

$$\langle 0|[Q_a, \Phi(x_1)]Y|0\rangle = -i\langle 0|G_a\Phi(x_1)Y|0\rangle \tag{94}$$

This being true for an arbitrary Y we conclude

$$[Q_a, \Phi(x_1)] = -iG_a\Phi(x_1) \tag{95}$$

In other words the conserved charge Q_a is the generator of the infinitesimal symmetry transformation in the operator formalism.

Integrating (92) over all space-time we obtain:

$$\delta_{\omega_a}\langle\Phi(x_1)\cdots\Phi(x_n)\rangle = -i\omega_a\sum_{i=1}^{n}\langle\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\rangle \tag{96}$$

In other words the variation of the correlator under an infinitesimal transformation vanishes.

Lecture 3
Conformal group in $d > 2$ dimensions

We denote by $g_{\mu\nu}$ the metric tensor in a space-time of dimension d. By definition a conformal transformation of the coordinates is an invertible mapping $x \to x'$ which leaves the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{97}$$

where

$$g'_{\mu\nu}(x') \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} = g_{\lambda\rho} \tag{98}$$

In other words, a conformal transformation is locally equivalent to rotation and a dilatation. For simplicity we assume that the conformal transformation is an infinitesimal deformation of the standard Cartesian metric $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(1, \ldots, 1)$.

The set of conformal transformations manifestly forms a group, and it obviously has the Poincaré group as a subgroup, since the latter corresponds to the special case $\Lambda(x) = 1$. Let us investigate the consequences of the definition (97) on an infinitesimal transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \tag{99}$$

It follows from (97)

$$\eta_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} = \Lambda^{-1} \eta_{\lambda\rho} \tag{100}$$

and inserting (99) we obtain in the first order by ϵ:

$$\eta_{\mu\nu} \left(\delta^{\mu}_{\lambda} + \frac{\partial \epsilon^{\mu}}{\partial x^{\lambda}} \right) \left(\delta^{\nu}_{\rho} + \frac{\partial \epsilon^{\nu}}{\partial x^{\rho}} \right) = \eta_{\lambda\rho} + \partial_{\lambda} \epsilon_{\rho} + \partial_{\rho} \epsilon_{\lambda} \tag{101}$$

Therefore the requirement that the transformation be conformal implies that

$$\partial_{\mu} \epsilon_{\nu} + \partial_{\nu} \epsilon_{\mu} = (\Lambda^{-1} - 1) \eta_{\mu\nu} = f(x) \eta_{\mu\nu} \tag{102}$$

The factor $f(\mathbf{x})$ is determined by taking the trace on both sides:

$$f(\mathbf{x}) = \frac{2}{d} \partial_{\rho} \epsilon^{\rho} \tag{103}$$

By applying an extra derivative ∂_{ρ} on Eq. (102), permuting the indices and taking a linear combination, we arrive at

$$2 \partial_{\mu} \partial_{\nu} \epsilon_{\rho} = \eta_{\mu\rho} \partial_{\nu} f + \eta_{\nu\rho} \partial_{\mu} f - \eta_{\mu\nu} \partial_{\rho} f \tag{104}$$

Upon contracting with $\eta_{\mu\nu}$ this becomes

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu f \tag{105}$$

Applying ∂_ν on this expression and ∂^2 on Eq. (102) we find

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f \tag{106}$$

Finally, contracting with $\eta_{\mu\nu}$ we end up with

$$(d-1)\partial^2 f = 0 \tag{107}$$

Now we can derive the explicit form of conformal transformation in d dimensions. First if $d = 1$ the above equations do not impose any constraint on the function f, and therefore any smooth transformation is conformal in one dimension. This is a trivial statement since notion of angle then does not exist. The case $d = 2$ will be studied in detail later. For the moment we concentrate on the case $d \geq 3$. Equations (106) and (107) imply that $\partial_\mu\partial_\nu f = 0$ (i.e. that the function is at most linear in the coordinates):

$$f(x) = A + B_\mu x^\mu \tag{108}$$

If we substitute this expression into (104) we see that $\partial_\mu\partial_\nu\epsilon_\rho$ is constant, which means that ϵ_μ is at most quadratic in the coordinates. We therefore write the general expression

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho, \quad c_{\mu\nu\rho} = c_{\nu\mu\rho} \tag{109}$$

Since the constraints above hold for all x we may treat each power of the coordinate separately. It follows that the constant term a_μ is free of coordinates.This term amounts to an infinitesimal translation. Substitution of the linear term into (102) yields

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\lambda_\lambda \eta_{\mu\nu} \tag{110}$$

which implies that $b_{\mu\nu}$ is the sum of an antisymmetric part and a pure trace:

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \qquad m_{\mu\nu} = -m_{\nu\mu} \tag{111}$$

The pure trace represents an infinitesimal scale transformation, whereas the antisymmetric part is an infinitesimal rigid rotation. Substitution of the quadratic term of (109) into (104) yields

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \qquad b_\mu = \frac{1}{d} c^\sigma_{\sigma\mu} \tag{112}$$

and the corresponding infinitesimal transformation is

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2 \tag{113}$$

which bears the name of special conformal transformation (SCT).

The finite transformations corresponding to the above are the following: Translation:

$$x'^\mu = x^\mu + a^\mu \tag{114}$$

Dilation:

$$x'^\mu = \alpha x^\mu \tag{115}$$

Rigid Rotation:

$$x'^\mu = M^\mu_\nu x^\mu \qquad M^\mu_\rho M^\nu_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \tag{116}$$

SCT:

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2x \cdot b + b^2 x^2} \tag{117}$$

Taking into account that generators are given by the first derivatives of the transformations (64) we can compute the generator corresponding to the parameter a via the formula:

$$X_a = -i \frac{\partial x'^\mu}{\partial \theta^a} \frac{\partial}{\partial x^\mu} \tag{118}$$

The formula for the infinitesimal translations

$$x'^\mu = x^\mu + a^\mu \tag{119}$$

implies for the generator of the translations

$$P_\mu = -i\partial_\mu \tag{120}$$

The formula for the infinitesimal scale transformation

$$x'^\mu = x^\mu + \alpha x^\mu \tag{121}$$

implies

$$\frac{\partial x'^\mu}{\partial \alpha} = x^\mu \tag{122}$$

and yields the generator of dilation

$$D = -ix^\mu \partial_\mu \tag{123}$$

For the infinitesimal rotation

$$x'^{\mu} = x^{\mu} + m^{\mu}_{\nu} x^{\nu} = x^{\mu} + m_{\rho\nu}\eta^{\rho\mu} x^{\nu} = x^{\mu} + \frac{1}{2} m_{\rho\nu}(\eta^{\rho\mu} x^{\nu} - \eta^{\nu\mu} x^{\rho}) \tag{124}$$

we have

$$\frac{\partial x'^{\mu}}{\partial m_{\rho\nu}} = \eta^{\rho\mu} x^{\nu} - \eta^{\nu\mu} x^{\rho} \tag{125}$$

Inserting this in (118 we get

$$L^{\rho\nu} = -i(x^{\rho}\partial^{\nu} - x^{\rho}\partial^{\mu}) \tag{126}$$

Finally for the infinitesimal special conformal transformation

$$x'^{\mu} = x^{\mu} + 2(x \cdot b)x^{\mu} - b^{\mu} x^2 \tag{127}$$

we obtain

$$\frac{\partial x'^{\mu}}{\partial b^{\nu}} = 2x_{\nu} x^{\mu} - \delta^{\mu}_{\nu} x^2 \tag{128}$$

and

$$K_{\nu} = -i(2x_{\nu} x^{\mu}\partial_{\mu} - x^2\partial_{\nu}) \tag{129}$$

Collecting all we have:

$$P_{\mu} = -i\partial_{\mu} \tag{130}$$

$$D = -ix^{\mu}\partial_{\mu} \tag{131}$$

$$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \tag{132}$$

$$K_{\mu} = -i(2x_{\mu} x^{\nu}\partial_{\nu} - x^2\partial_{\mu}) \tag{133}$$

These generators obey the following commutation rules, which in fact define the conformal algebra:

$$[D, P_{\mu}] = iP_{\mu} \tag{134}$$

$$[D, K_{\mu}] = -iK_{\mu} \tag{135}$$

$$[K_{\mu}, P_{\nu}] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \tag{136}$$

$$[K_{\rho}, L_{\mu\nu}] = i(\eta_{\rho\mu} K_{\nu} - \eta_{\rho\nu} K_{\mu}) \tag{137}$$

$$[P_{\rho}, L_{\mu\nu}] = i(\eta_{\rho\mu} P_{\nu} - \eta_{\rho\nu} P_{\mu}) \tag{138}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\rho\nu} L_{\mu\sigma} + \eta_{\sigma\mu} L_{\nu\rho} - \eta_{\rho\mu} L_{\nu\sigma} - \eta_{\sigma\nu} L_{\mu\rho}) \tag{139}$$

$$[P_{\mu}, P_{\nu}] = 0 \tag{140}$$

$$[K_{\mu}, K_{\nu}] = 0 \tag{141}$$

$$\tag{142}$$

In order to put the above commutation rules into a simpler from, we define the following generators:

$$J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1,\nu} = \frac{1}{2}(P_\mu - K_\mu) \tag{143}$$

$$J_{-1,0} = D \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{144}$$

$$\tag{145}$$

where $J_{ab} = -J_{ba}$ and $a, b \in \{-1, 0, 1, \ldots, d\}$. These new generators obey the $SO(d+1, 1)$ commutation relations:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{146}$$

where the diagonal metric η_{ab} is $\text{diag}(-1, 1, 1, \ldots, 1)$. This shows the isomorphism between the conformal group in d dimensions and the group $SO(d+1, 1)$ with $\frac{1}{2}(d+2)(d+1)$ parameters.

Let us now generalize the formulas (**??**) to fileds with internal quantum numbers.

We start by studying the subgroup of the Poincare group that leaves th point $x = 0$ invariant, that is the Lorentz group. We then introduce a matrix representation $S^{\mu\nu}$ to define the action of infinitesimal Lorentz transformation on the field

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{147}$$

$S_{\mu\nu}$ is the spin operator associated with the field Φ. Next,by use of commutation relations on the Poincare group, we translate the generator $L_{\mu\nu}$ to a nonzero value pf x

$$e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{148}$$

The above translation is explicitly calculated by use of the Hausdorff formula

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A].A] + \cdots \tag{149}$$

This can be proved noting that

$$\frac{d}{dt}(e^{-tA}Be^{tA}) = e^{-tA}[B, A]e^{tA} \tag{150}$$

Repeatedly using this relation we see that higher derivatives are given by the repeated commutators and (149) is the Tailor expansion at the value $t = 1$. This allows us to write the action of the generators:

$$P_\mu\Phi(x) = -i\partial_\mu\Phi(x) \tag{151}$$

$$L_{\mu\nu}\Phi(x) = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + S_{\mu\nu}\Phi(x) \tag{152}$$

We proceed in the same way for the full conform group. The subgroup that leaves the origin $x = 0$ invariant is generated by rotations, dilations, and special conformal transformations. We denote by $S_{\mu\nu}$, Δ and κ_μ the respective values of the generators $L_{\mu\nu}$, D, and K_μ at $x = 0$. The commutations (134) then allows us to translate the generators, using the Hausdorff formula (149)

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \tag{153}$$

$$e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu(x^\nu P_\nu) - x^2 P_\mu \tag{154}$$

from which we arrive finally at the following extra transformation rules:

$$D\Phi(x) = (-ix^\nu\partial_\nu + \Delta)\Phi(x) \tag{155}$$

$$K_\mu\Phi(x) = \left(\kappa_\mu + 2x_\mu\Delta - 2x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu\right)\Phi(x) \tag{156}$$

It is possible to show that $\kappa_\mu = 0$.

In principle, we can derive from the above the change of $\Phi(x)$ under a finite conformal transformation. However, we shall give the result only for spinless fields $S_{\mu\nu} = 0$. Under a conformal transformation $x \to x'$ a spinless fields $\phi(x)$ transform as

$$\phi(\mathbf{x}) \to \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta/d} \phi(\mathbf{x}') \tag{157}$$

where $\left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|$ is the Jacobian of the conformal transformation of the coordinates. Computing determinant from the both sides of (100) we obtain that Jacobian is related to the scale factor of the metric :

$$\left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right| = \Lambda(x)^{-d/2} \tag{158}$$

Lecture 4
Correlation functions

We define a theory with conformal invariance to satisfy the properties:

1. There is a set of fields $\{A_i\}$, where the index i specifies the different fields. This set of fields in general is infinite and contains in particular the derivatives of all the fields.

2. There is a subset of fields $\{\phi_j\} \in \{A_i\}$, called quasi-primary that under global conformal transformations transform according to

$$\phi_j(\mathbf{x}) \to \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_j/d} \phi_j(\mathbf{x}') \tag{159}$$

where $\left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|$ is the Jacobian of the conformal transformation of the coordinates.

3. The rest of the $\{A_i\}$ can be expressed as linear combinations of the quasi-primary fields and their derivatives.

4. There is a vacuum $|0\rangle$ invariant under the global conformal group

$$U|0\rangle = |0\rangle \tag{160}$$

This implies

$$\langle 0|U^{-1}\phi_1 U \cdots U^{-1}\phi_n U|0\rangle = \langle 0|\phi_1 \cdots \phi_n|0\rangle \tag{161}$$

Taking into account that

$$U^{-1}\phi_j(\mathbf{x})U = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_j/d} \phi_j(\mathbf{x}') \tag{162}$$

we obtain

$$\langle \phi_1(\mathbf{x}_1)\ldots\phi_n(\mathbf{x}_n)\rangle = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_1}^{\Delta_1/d} \cdots \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_n}^{\Delta_n/d} \langle \phi_1(\mathbf{x}_1')\ldots\phi_n(\mathbf{x}_n')\rangle \tag{163}$$

Let us now compute the two-point correlation function of the quasi-primary fields. According to (163) two-point functions have the following transformation rule:

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_1}^{\Delta_1/d} \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_2}^{\Delta_2/d} \langle \phi_1(\mathbf{x}_1')\phi_2(\mathbf{x}_2')\rangle \tag{164}$$

If we specialize to a scale transformation $\mathbf{x} \to \lambda \mathbf{x}$ we obtain:

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda\mathbf{x}_1)\phi_2(\lambda\mathbf{x}_2)\rangle \tag{165}$$

Rotation and translation invariance require that:

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = f(|\mathbf{x}_1 - \mathbf{x}_2|) \tag{166}$$

where $f(\mathbf{x}) = \lambda^{\Delta_1+\Delta_2} f(\lambda\mathbf{x})$ by virtue of (165). In other words

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \frac{C}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}} \tag{167}$$

where C is a constant coefficient. It remains to use the invariance under special conformal transformation. Recalling that for such a transformation

$$\left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right| = \frac{1}{(1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2)^d} \tag{168}$$

and the transformation property of the distance under the special conformal transformations:

$$|\mathbf{x}'_1 - \mathbf{x}'_2| = \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{(1 - 2\mathbf{b}\cdot\mathbf{x}_1 + b^2\mathbf{x}_1^2)^{\frac{1}{2}}(1 - 2\mathbf{b}\cdot\mathbf{x}_2 + b^2\mathbf{x}_2^2)^{\frac{1}{2}}} \tag{169}$$

the covariance of the correlation function (167) implies

$$\frac{C}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}} = \frac{C}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}} \tag{170}$$

This constraint is identically satisfied only if $\Delta_1 = \Delta_2$. In other words two quasy-primary fields correlated only if they have the same scaling dimensions:

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = 0, \qquad \text{if} \qquad \Delta_1 \neq \Delta_2 \tag{171}$$

and

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \frac{C}{|\mathbf{x}_1 - \mathbf{x}_2|^{2\Delta_1}} \qquad \text{if} \qquad \Delta_1 = \Delta_2 \tag{172}$$

A similar analysis may be performed on three-point functions. Covariance under rotations, translations and dilatations forces a generic three-point function to have the following form

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\phi_3(\mathbf{x}_3)\rangle = \frac{C}{x_{12}^a x_{23}^b x_{13}^c} \tag{173}$$

where $x_{ij} = |x_i - x_j|$ and with a, b, c such that

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \tag{174}$$

Under special conformal transformations Eq.(173) becomes

$$\frac{C}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{13}^c} \tag{175}$$

For this expression to be of the same form as Eq.(173) all the factors involving parameter b^μ must disappear, which leads to the following set of constraints:

$$a + c = 2\Delta_1, \qquad a + b = 2\Delta_2, \qquad b + c = 2\Delta_3 \tag{176}$$

The solution to to these constraints is unique:

$$a = \Delta_1 + \Delta_2 - \Delta_3 \tag{177}$$

$$b = \Delta_2 + \Delta_3 - \Delta_1 \tag{178}$$

$$c = \Delta_1 + \Delta_3 - \Delta_2 \tag{179}$$

$$\tag{180}$$

Therefore the three-point correlator is

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\phi_3(\mathbf{x}_3)\rangle = \frac{C}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}} \tag{181}$$

Lecture 5
Conformal group in two dimensions

Let us consider the conformal transformations in two dimensions $D = 2$. Condition (102) becomes the Cauchy-Riemann equation

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \tag{182}$$

It is then natural to write $\epsilon(z) = \epsilon_1 - i\epsilon_2$ and $\bar{\epsilon}(\bar{z}) = \epsilon_1 + i\epsilon_2$ in the complex coordinates $z = x+iy$ and $\bar{z} = x-iy$. Two dimensional conformal transformations thus coincide with the analytic coordinate transformations

$$z \to f(z) \qquad\qquad \bar{z} \to f(\bar{z}) \tag{183}$$

The metric in the complex coordinates is

$$ds^2 = dzd\bar{z} \tag{184}$$

Under the analytic coordinate transformations

$$z \to f(z) \qquad\qquad \bar{z} \to f(\bar{z}) \tag{185}$$

$$ds^2 = dzd\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z} \tag{186}$$

Thus the group of two-dimensional conformal transformations coincides with the analytic coordinate transformations. Any holomorphic infinitesimal transformation may be expressed as:

$$z' = z + \epsilon(z) \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1} \tag{187}$$

The effect of such a mapping on a field $\phi(z, \bar{z})$ living on the plane is:

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_n \{c_n l_n \phi(z,\bar{z}) + \bar{c}_n \bar{l}_n \phi(z,\bar{z})\} \tag{188}$$

where we have introduced the generators

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{189}$$

These generators obey the following commutation relations:

$$[l_n, l_m] = (n-m)l_{n+m} \tag{190}$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m} \tag{191}$$

$$[l_n, \bar{l}_m] = 0 \tag{192}$$

$$\tag{193}$$

Thus the conformal algebra is the direct sum of two isomorphic algebras, each with very simple commutation relations. The algebra (190) is sometimes called the de Witt algebra.

Note that $l_0 = -z\partial_z$ and $\bar{l}_0 = -\bar{z}\partial_{\bar{z}}$ and hence introducing the polar coordinates $z = re^{i\theta}$ we obtain

$$r\frac{\partial}{\partial r} = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = -(l_0 + \bar{l}_0) \tag{194}$$

and

$$\frac{\partial}{\partial \theta} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} = -i(l_0 - \bar{l}_0) \tag{195}$$

Thus $(l_0 + \bar{l}_0)$ generates dilatations and $i(l_0 - \bar{l}_0)$ generates rotations.

Let us look for generators well-defined globally on the Riemann sphere $S^2 = C \cup \infty$. Holomorphic conformal transformations are generated by vector fields:

$$v(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1}\partial_z \tag{196}$$

Non-singularity of $v(z)$ as $z \to 0$ allows $a_n \neq 0$ only for $n \geq -1$.To investigate the behavior of $v(z)$ as $z \to \infty$, we perform the transformation $z = -\frac{1}{\omega}$,

$$v(z) = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n+1} \left(\frac{dz}{d\omega}\right)^{-1} \partial_\omega = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n-1} \partial_\omega \tag{197}$$

Non-singularity as $\omega \to 0$ allows $a_n \neq 0$ only for $n \leq 1$. We see that only the conformal transformations generated by $a_n l_n$ for $n = 0, \pm 1$ are globally defined. The same considerations apply to anti-holomorphic transformations.

These generators satisfy the commutation relation:

$$[l_0, l_{-1}] = l_{-1} \tag{198}$$

$$[l_0, l_1] = -l_1 \tag{199}$$

$$[l_1, l_{-1}] = 2l_0 \tag{200}$$

$$\tag{201}$$

and similar for antiholomorphic components.

This is precisely the $SU(2)$ rotation algebra if we identify l_0 with J_z, il_1 with $J^- = J_x - iJ_y$ and il_{-1} with $J^+ = J_x + iJ_y$.

Hence Lie algebra of the global conformal transformation consists of two commuting copies of the $SU(2)$ algebra, and therefore coincide with the naively expected conformal group in two dimensions $SO(3,1)$.

The fact that the Lie algebra of $SO(3,1)$ as well isomorphic to two commuting copies of $SU(2)$ algebra can be seen by means of identifications:

$$\mathbf{A} = (J^{23}, J^{31}, J^{12}) \tag{202}$$

$$\mathbf{B} = (J^{10}, J^{20}, J^{30}) \tag{203}$$

and then taking

$$\mathbf{J} = \frac{1}{2}(\mathbf{A} + i\mathbf{B}) \tag{204}$$

$$\bar{\mathbf{J}} = \frac{1}{2}(\mathbf{A} - i\mathbf{B}) \tag{205}$$

It is straightforward to check that $SO(3,1)$ commutation relations imply that $\mathbf{J}$ and $\bar{\mathbf{J}}$ provide two commuting copies of the $SU(2)$ algebra. One can see this also on the group level. We identify l_{-1} and $\bar{l}_{-1}$ as generators of translations (globally $z \to z + \alpha$), l_0 and $\bar{l}_0$ as generators of dilatations (globally $z \to \lambda z$), and l_1 and $\bar{l}_1$ as generators of special conformal transformations (globally $z \to \frac{1}{1-\beta z}$. The combined form of these transformations is

$$z \to \frac{az+b}{cz+d} \qquad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \tag{206}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. This is the group $SL(2,\mathbb{C})/\mathbb{Z}_2$. The quotient by $\mathbb{Z}_2$ is due to the fact that (1010) is unaffected by taking all of $a, b.c, d$ to minus of themselves. It remains to show that the quotient $SL(2,\mathbb{C})/\mathbb{Z}_2$ is isomorphic to the Lorentz group $SO(3,1)$.

For this purpose we organize the four-vector X^μ as hermitian matrix

$$X^\mu \sigma^\mu = \begin{pmatrix} X^0 + X^3 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - X^3 \end{pmatrix} \tag{207}$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and $\sigma_0 \equiv I$. Note that

$$\det\left(X^\mu \sigma^\mu\right) = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 \tag{208}$$

The transformation

$$M\left(X^\mu \sigma^\mu\right) M^\dagger \tag{209}$$

where $M \in SL(2,\mathbb{C})$ leaves the matrix hermitian and does not change the determinant. Therefore this transformation induces the Lorentz transformation of X^μ:

$$M\left(X^\mu \sigma^\mu\right) M^\dagger = (\Lambda(M)^\mu_\nu X^\nu \sigma^\mu) \tag{210}$$

where $\Lambda(M)^\mu_\nu \in SO(3,1)$. Since the map (209) is the same for M and $-M$ therefore $\Lambda(M)^\mu_\nu = \Lambda(-M)^\mu_\nu$ and we proved the isomorphism $SL(2,\mathbb{C})/\mathbb{Z}_2 \approx SO(3,1)$.

Lecture 6
Examples of CFT: Non-linear sigma model

Weyl transformation

$$g_{\mu\nu} \to \Lambda g_{\mu\nu} \tag{211}$$

Remembering that

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{212}$$

where $g = \det||g_{\mu\nu}||$, we obtain that the action is invariant under the Weyl transformation if the tensor-energy momentum is traceless

$$T^{\mu\nu} g_{\mu\nu} = 0 \tag{213}$$

On the other hand under an arbitrary transformation of the coordinates $x^\mu \to x^\mu + \epsilon^\mu$, the action changes as follows:

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{214}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor. The definition (102) of an infinitesimal conformal transformation implies that the corresponding variation of the action is

$$\delta S = \frac{1}{d} \int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho \tag{215}$$

The tracelessness of the energy-momentum tensor then implies the invariantness of the action under conformal transformation.

Consider now the action

$$S = \int_\Sigma d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) \tag{216}$$

where $h_{\alpha\beta}$ is metric on two-dimensional world-sheet Σ, $g_{\mu\nu}(X)$ is metric on manifold $\mathbf{M}$. It is Weyl invariant: under $h_{\alpha\beta} \to \Lambda h_{\alpha\beta}$ we have in d dimension:

$$\sqrt{h} h^{\alpha\beta} \to \Lambda^{\frac{d}{2}-1} \sqrt{h} h^{\alpha\beta} \tag{217}$$

and therefore the action is indeed Weyl invariant in two dimensions.

The tensor energy-momentum is

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) - \frac{1}{2} h_{\alpha\beta} h^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu g_{\mu\nu}(X) \tag{218}$$

and obviously traceless in two-dimensional case:

$$h^{\alpha\beta}T_{\alpha\beta} = 0 \tag{219}$$

Let us check if the Weyl invariant is not broken by the quantum corrections.

We will work in the dimensional regularization scheme. In $d = 2+\epsilon$ dimensions choosing conformal gauge

$$h_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \tag{220}$$

we obtain

$$S = \int_{\Sigma} d^2\sigma e^{\epsilon\phi}\partial_\alpha X^\mu \partial^\alpha X^\nu g_{\mu\nu}(X) \tag{221}$$

Let expand the field X^μ around constant solution, a point X_0^μ

$$X^\mu(\sigma,\tau) = X_0^\mu + x^\mu(\sigma,\tau) \tag{222}$$

where $x^\mu(\sigma,\tau)$ quantum fluctuations. Choosing normal Riemann coordinates we have for metric:

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\kappa}x^\lambda x^\kappa - \frac{1}{6}D_\lambda R_{\mu\lambda\nu\kappa}x^\rho x^\lambda x^\kappa + \cdots \tag{223}$$

where $R_{\mu\lambda\nu\kappa}$ Riemann tensor on manifold $\mathbf{M}$ in point X_0^μ. Inserting this in action and also expanding $e^{\epsilon\phi} = 1 + \epsilon\phi \cdots$ we get

$$S = \int_{\Sigma} d^2\sigma \Big[\partial_\alpha x^\mu \partial^\alpha x^\nu \eta_{\mu\nu}(1+\epsilon\phi) - \frac{1}{3}R_{\mu\lambda\nu\kappa}x^\lambda x^\kappa \partial_\alpha x^\mu \partial^\alpha x^\nu (1+\epsilon\phi) + \cdots\Big] \tag{224}$$

Consider the contraction

$$\langle x^\lambda(\sigma) x^\kappa(\sigma')\rangle_{\sigma\to\sigma'} = \pi\eta_{\lambda\kappa}\mathrm{lim}_{\sigma\to\sigma'} \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{e^{ik(\sigma-\sigma')}}{k^2} \tag{225}$$

Let us compute the integral

$$I(D) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{p^2+m^2} \tag{226}$$

In polar coordinates it can be written:

$$I(D) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{p^2+m^2} = \frac{S_D}{(2\pi)^D}\int_0^\infty dp p^{D-1}\frac{1}{p^2+m^2} \tag{227}$$

where

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{228}$$

is the surface of a unit sphere in D dimensions. The resulting one-dimensional integral can after the substitution $p^2 = ym^2$, be cast into the form of an integral for the Beta function

$$B(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} = \int_0^\infty dy y^{\alpha-1}(1+y)^{-\alpha-\gamma} \tag{229}$$

We then find

$$I(D) = \frac{S_D}{(2\pi)^D}\int_0^\infty dp p^{D-1}\frac{1}{p^2+m^2} = \frac{S_D}{2(2\pi)^D}(m^2)^{D/2-1}\int_0^\infty dy y^{D/2-1}(1-y)^{-1} = \tag{230}$$
$$\frac{1}{(4\pi)^{D/2}\Gamma(D/2)}(m^2)^{D/2-1}\frac{\Gamma(D/2)\Gamma(1-D/2)}{\Gamma(1)} = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}}\Gamma(1-D/2)$$

Using this in the limit $D = \lim_{\epsilon\to 0}(2+\epsilon)$ and remembering

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} \tag{231}$$

we obtain

$$\langle x^\lambda(\sigma)x^\kappa(\sigma')\rangle_{\sigma\to\sigma'} \sim \frac{\eta^{\lambda\kappa}}{2\epsilon} \tag{232}$$

This implies that one-loop correction to the metric resulting from the curvature term is

$$\int_\Sigma d^2\sigma\phi(\sigma)\partial_\alpha x^\mu\partial^\alpha x^\nu R_{\mu\nu}(X_0^\rho) \tag{233}$$

Here $R_{\mu\nu}(X_0^\rho)$ is Ricci tensor on manifold $\mathbf{M}$. Therefore in general the Weyl invariance is broken, since in the limit $\epsilon \to 0$ the scale factor $\phi(\sigma)$ is remained. We obtain that the sigma-model action is conformal invariant for Ricci flat manifold $\mathbf{M}$.

One can consider also sigma model with the B- term

$$S = \int_\Sigma d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X) \tag{234}$$

Similar calculations bring to the following condition of the Weyl one-loop invariance

$$R_{\mu\nu} + \frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho} = 0 \tag{235}$$

where $H = dB$.

Lecture 7
Tensor energy-momentum, radial quantization, OPE

Under an arbitrary transformation of the coordinates $x^\mu \to x^\mu + \epsilon^\mu$, the action changes as follows:

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \tag{236}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor. The definition (102) of an infinitesimal conformal transformation implies that the corresponding variation of the action is

$$\delta S = \frac{1}{d} \int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho \tag{237}$$

The tracelessness of the energy-momentum tensor then implies the invariantness of the action under conformal transformation.

The current of conformal symmetry is

$$J_\mu = T_{\mu\nu} \epsilon^\nu \tag{238}$$

This current is conserved because

$$\partial^\mu J_\mu = \partial^\mu T_{\mu\nu} \epsilon^\nu + T_{\mu\nu} \partial^\mu \epsilon^\nu = 0 \tag{239}$$

which vanishes because the tensor energy-momentum is conserved and traceless.

To implement the conservation equations in the complex plane we compute the components of tensors in the complex coordinates. Since the flat Euclidean metric $ds^2 = dx^2 + dy^2$ in the complex coordinates $z = x + iy$ has the form $ds^2 = dz d\bar{z}$ one has

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{240}$$

and

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} \tag{241}$$

and

$$g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{242}$$

and

$$g^{z\bar{z}} = g^{\bar{z}z} = 2 \tag{243}$$

The components of the energy-momentum tensor in this frame are

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \tag{244}$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \tag{245}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^{\mu}_{\mu} \tag{246}$$

$$\tag{247}$$

Therefore the tracelessness implies

$$T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{248}$$

The conservation law $g^{\alpha\mu}\partial_\alpha T_{\mu\nu} = 0$ gives two relations

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \tag{249}$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0$$

Using (248) we obtain

$$\partial_{\bar{z}} T_{zz} = 0 \qquad \text{and} \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{250}$$

The two non-vanishing components of the energy-momentum tensor

$$T(z) \equiv T_{zz}(z) \qquad \text{and} \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \tag{251}$$

thus have only holomorphic and anti-holomorphic dependences.

Consider the system on a cylinder $\Sigma = R \times S^1 = (t, x \mod 2\pi)$, where t is world-sheet time, and x is compactified space coordinate.

Next we consider the conformal map $w \to z = e^w = e^{t+ix}$, that maps the cylinder to the complex plane. Then infinite past and future on the cylinder, $t = \pm\infty$ are mapped to the points $z = 0, \infty$ on the plane. Equal time surfaces, $t =$ const becomes circles of constant radius on the z-plane. Dilatation on the plane e^a becomes time translation $t + a$ on the cylinder, and rotation on the plane $e^{i\alpha}$ is space translation $x + \alpha$ on the cylinder. Therefore the dilatation generator on the conformal plane can be regarded as the Hamiltonian, and the rotation generator on the conformal plane can be regarded as momentum.

The current of conformal transformations takes the form:

$$J_z = T(z)\epsilon(z) \tag{252}$$

$$J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$$

The conserved charge of the conformal transformations takes the form

$$Q = \frac{1}{2\pi i}\oint dz T(z)\epsilon(z) + \frac{1}{2\pi i}\oint d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{253}$$

Radial ordering

Product of operators make sense if they are radially ordered. This is the analogue of time ordering for field theory on the cylinder. In the classical theory the ordering of fields or charges in a product is of course irrelevant. In the quantum theory they become operators and we have to specify an ordering. The product of two operators $A(x_a, t_a)$ and $B(x_b, t_b)$ can be written with the help of the Hamiltonian H of the system as

$$A(x_a, t_a)B(x_b, t_b) = e^{iHt_a}A(x_a, 0)e^{-iHt_a}e^{iHt_b}B(x_b, 0)e^{-iHt_b} \tag{254}$$

The factor $e^{-iH(t_a - t_b)}$ becomes $e^{-H(\tau_a - \tau_b)}$ when we Wick rotate. Usually the Hamiltonian is bounded from below, but not from above. Then if $\tau_a < \tau_b$ the exponential can take arbitrarily large values, and expectation values of the operator product are then not defined. Hence in operator product one imposes time ordering, denoted as

$$TA(t_a)B(t_b) = A(t_a)B(t_b) \text{ for } t_a > t_b \text{ and } B(t_b)A(t_a) \text{ for } t_a < t_b \tag{255}$$

After mapping from the cylinder to the plane, the Euclidean time coordinate is mapped to the radial coordinate, and time ordering becomes radial ordering

$$RA(z)B(w) = A(z)B(w) \text{ for } |z| > |w| \text{ and } B(w)A(z) \text{ for } |z| < |w| \tag{256}$$

The variation of any field is given by commutator with the charge (253):

$$\begin{aligned} &\delta_{\epsilon,\bar{\epsilon}}\Phi(w, \bar{w}) = [Q, \Phi(w, \bar{w})] = \\ &\frac{1}{2\pi i}\oint dz\epsilon(z)(T(z)\Phi(w, \bar{w}) - \Phi(w, \bar{w})T(z)) + \\ &\frac{1}{2\pi i}\oint d\bar{z}\bar{\epsilon}(\bar{z})(\bar{T}(\bar{z})\Phi(w, \bar{w}) - \Phi(w, \bar{w})\bar{T}(\bar{z})) \end{aligned} \tag{257}$$

Let us now analyze the order of operators in the second and the third lines in (257). We will discuss the holomorphic part, the similar discussion holds for antiholomorphic part. We have seen that the first term in the commutator is defined only if $|z| > |w|$, whereas the second one requires $|z| < |w|$. Note however that z is an integration variable, and that the definition of Q did not include any prescription for the precise contours to be used. Classically $Q$is in fact independent of the contour due to Cauchy's theorem, because the integrand is a holomorphic function. On the cylinder this can be interpreted as charge conservation, *i.e.* evaluated Q at two different times gives the same answer. In the quantum theory we have to be more careful. As one usually does, we use the freedom we have in the classical theory in order to write the quantity on interests in such a way that it is well-defined after quantization. Nothing forbids us to use different contours in two terms in commutator:

$$\begin{aligned}&\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \\ &\frac{1}{2\pi i}\oint_{|z|>|w|} dz\epsilon(z)(T(z)\Phi(w,\bar{w}) - \oint_{|z|<|w|}\Phi(w,\bar{w})T(z))\end{aligned} \tag{258}$$

Using the definition of the radial ordering (256) one can write

$$\begin{aligned}&\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \\ &\frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\epsilon(z)R(T(z)\Phi(w,\bar{w}))\end{aligned} \tag{259}$$

Deforming the contours the result is

$$\begin{aligned}&\frac{1}{2\pi i}\oint dz\epsilon(z)[T(z),\Phi(w,\bar{w})] = \\ &\frac{1}{2\pi i}\oint_{w} dz\epsilon(z)R(T(z)\Phi(w,\bar{w}))\end{aligned} \tag{260}$$

where the integration contour encircles the point w. Collecting all we obtain:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{w} dz\epsilon(z)R(T(z)\Phi(w,\bar{w})) + \frac{1}{2\pi i}\oint_{w} d\bar{z}\bar{\epsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(w,\bar{w})) \tag{261}$$

Primary fields possess the following transformation rule:

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^{h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}}\Phi(f(z),\bar{f}(\bar{z})) \tag{262}$$

The infinitesimal transformation of the primary fields of the weight h and $\bar{h}$ is:

$$\begin{aligned}&\delta_{\epsilon,\bar{\epsilon}}\Phi(w,\bar{w}) = h\partial\epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial\Phi(w,\bar{w}) + \\ &\bar{h}\bar{\partial}\bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\bar{\partial}\Phi(w,\bar{w})\end{aligned} \tag{263}$$

Comparing (260) and (263) we get OPE of the energy-momentum tensor with the primary field of the weights h, $\bar{h}$

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w}) \tag{264}$$

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\Phi(w,\bar{w}) \tag{265}$$

Lecture 8
Virasoro algebra

Schwarzian derivative

OPE of the tensor energy-momentum with itself takes the form:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{266}$$

The term on the rhs, with coefficient c a constant, is allowed by analicity, Bose symmetry, and scale invariance. Apart from this term, (279) is just the statement that $T(z)$ is a conformal field of weight $(2,0)$. According to (261) the variation of T under infinitesimal conformal transformation is

$$\delta_\epsilon T(w) = \frac{1}{2\pi i}\oint \epsilon(z)T(z)T(w) \tag{267}$$

$$\frac{1}{12}c\partial_w^3\epsilon(w) + 2T(w)\partial_w\epsilon(w) + \epsilon(w)\partial_w T(w)$$

The exponentiation of this infinitesimal variation to a finite transformation $z \to w(z)$ is

$$T(z) \to \left(\frac{dw}{dz}\right)^2 T(w(z)) + \frac{c}{12}S(w;z) \tag{268}$$

where we have introduced the Schwarzian derivative:

$$S(w;z) = \frac{(d^3w/dz^3)}{(dw/dz)} - \frac{3}{2}\left(\frac{(d^2w/dz^2)}{(dw/dz)}\right)^2 \tag{269}$$

It is the unique weight two object that vanishes when restricted to the global $SL(2,C)$ subgroup of the two-dimensional group. It also satisfies the composition law:

$$S(w,z) = \left(\frac{df}{dz}\right)^2 S(w,f) + S(f,z) \tag{270}$$

The tensor energy-momentum is thus example of a field that is quasi-primary, i.e. $SL(2,C)$ primary, but not Virasoro primary. For the exponential map $w \to z = e^w$ we have

$$S(e^w, w) = -1/2 \tag{271}$$

so

$$T_{\rm cyl}(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}S(z,w) = z^2T(z) - \frac{c}{24} \tag{272}$$

Substituting the mode expansion $T(z) = \sum L_n z^{-n-2}$ we find

$$T_{\text{cyl}}(w) = \sum L_n z^{-n} - \frac{c}{24} = \sum_n \left(L_n - \frac{c}{24}\delta_{n0} \right) e^{-nw} \tag{273}$$

The translation generator $(L_0)_{\text{cyl}}$ on the cilinder is thus given in terms of the dilatation generator L_0 on ther plane as

$$(L_0)_{\text{cyl}} = L_0 - \frac{c}{24} \tag{274}$$

Virasoro generators

We introduced a current $J(z) = T(z)\epsilon(z)$. Since $\epsilon(z)$ is an arbitrary holomorphic function, it is natural to expand it in modes. We expect that the current $T(z)z^{n+1}$ generates the transformation $z \to z + c_n z^{n+1}$. The corresponding charges are:

$$L_n = \frac{1}{2\pi i} \oint dz T(z) z^{n+1} \tag{275}$$

This relation can be inverted:

$$T(z) = \sum_n z^{-n-2} L_n \tag{276}$$

The commutator of the charges is

$$\begin{aligned}
&[L_n, L_m] = \qquad (277)\\
&\frac{1}{(2\pi i)^2} \oint_0 dw w^{m+1} \oint_w dz z^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) \right] = \\
&\frac{1}{2\pi i} \oint_0 dw w^{m+1} \left[\frac{1}{12} c(n+1)n(n-1) w^{n-2} + 2(n+1) w^n T(w) + w^{n+1} \partial T(w) \right] = \\
&\frac{1}{12} cn(n^2-1)\delta_{n+m,0} + 2(n+1) L_{m+n} - \frac{1}{2\pi i} \oint_0 dw (n+m+2) w^{n+m+1} T(w) = \\
&\frac{1}{12} cn(n^2-1)\delta_{n+m,0} + (n-m) L_{m+n}
\end{aligned}$$

Here we used

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z-z_0)^{n+1}} \tag{278}$$

Identical consideration for $\bar{T}$ implies

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2} \bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial \bar{T}(\bar{w}) \tag{279}$$

$$\bar{T}(\bar{z}) = \sum_n \bar{z}^{-n-2} \bar{L}_n \tag{280}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{1}{12}\bar{c}n(n^2-1)\delta_{n+m,0} \tag{281}$$

Since $T(z)$ and $\bar{T}(\bar{z})$ have no power law singularity in their product, we have

$$[L_n, \bar{L}_m] = 0 \tag{282}$$

Correlation functions

Since global conformal group $SL(2,C)$ preserves vacuum and anomaly free we have for $f(z)$ in the form (206):

$$\langle \Phi_1(z_1,\bar{z}_1)\dots\Phi_n(z_n,\bar{z}_n)\rangle = \prod_j \left(\partial f(z_j)\right)^{h_j} \left(\bar{\partial}\bar{f}(\bar{z}_j)\right)^{\bar{h}_j} \langle \Phi_1(f(z_1),\bar{f}(\bar{z}_1))\dots\Phi_n(f(z_n),\bar{f}(\bar{z}_n))\rangle \tag{283}$$

$$f(z) = \frac{az+b}{cz+d} \tag{284}$$

$$f'(z) = \frac{1}{(cz+d)^2} \tag{285}$$

$$f(z_1) - f(z_2) = \frac{z_1 - z_2}{(cz_1+d)(cz_2+d)} \tag{286}$$

Two-point function is

$$\langle \Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\rangle = \frac{C}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \tag{287}$$

where $h_1 = h_2 = h$, $\bar{h}_1 = \bar{h}_2 = \bar{h}$

$$\langle \Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \tag{288}$$

$$\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

where $z_{ij} = z_i - z_j$.

We can express the invariance rule (283) in infinitesimal form:

$$\sum_{i=1}^{n} \partial_i \langle \Phi_1(z_1,\bar{z}_1)\dots\Phi_n(z_n,\bar{z}_n)\rangle = 0 \tag{289}$$

$$\sum_{i=1}^{n} (z_i\partial_i + h_i) \langle \Phi_1(z_1,\bar{z}_1)\dots\Phi_n(z_n,\bar{z}_n)\rangle = 0 \tag{290}$$

$$\sum_{i=1}^{n} (z_i^2\partial_i + 2z_ih_i) \langle \Phi_1(z_1,\bar{z}_1)\dots\Phi_n(z_n,\bar{z}_n)\rangle = 0 \tag{291}$$

In- and Out- states

We suggest

$$[\Phi(z,\bar{z})]^{\dagger} = \Phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\frac{1}{\bar{z}^{2h}}\frac{1}{z^{2\bar{h}}} \tag{292}$$

To justify the ansatz (292) we consider in- and out- states. Since time $t \to -\infty$ on the cylinder corresponds to the origin of the z-plane, it is natural to define in-states as

$$|\Phi_{\rm in}\rangle \equiv \lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})|0\rangle \tag{293}$$

To define $\langle\Phi_{\rm out}|$ we need to construct the analogous object for $z \to \infty$. If we call $\tilde{\Phi}(w,\bar{w})$ the operator in the coordinates for which $w \to 0$ correspond to the point at ∞, then the natural definition is

$$\langle\Phi_{\rm out}| \equiv \lim_{w,\bar{w}\to 0}\langle 0|\tilde{\Phi}(w,\bar{w}) \tag{294}$$

We need to relate $\tilde{\Phi}(w,\bar{w})$ to $\Phi(z,\bar{z})$. Recall that for primary fields we have under $w \to f(w)$

$$\tilde{\Phi}(w,\bar{w}) = \Phi(f(w),\bar{f}(\bar{w}))(\partial f(w))^{h}(\bar{\partial}\bar{f}(\bar{w}))^{\bar{h}} \tag{295}$$

so that in particular under $f(w) \to 1/w$

$$\tilde{\Phi}(w,\bar{w}) = \Phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)(-w^{-2})^{h}(-\bar{w}^{-2})^{\bar{h}} \tag{296}$$

Demanding $\langle\Phi_{\rm out}| = |\Phi_{\rm in}\rangle^{\dagger}$ we arrive to (292).

$$\langle\Phi_{\rm out}| = \lim_{w,\bar{w}\to 0}\langle 0|\tilde{\Phi}(w,\bar{w}) = \lim_{w,\bar{w}\to 0}\langle 0|\Phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)\frac{1}{w^{2h}}\frac{1}{\bar{w}^{2\bar{h}}} \tag{297}$$

$$\lim_{w,\bar{w}\to 0}\langle 0|[\Phi(\bar{w},w)]^{\dagger} = \left[\lim_{w,\bar{w}\to 0}\Phi(\bar{w},w)|0\rangle\right]^{\dagger} = |\Phi_{\rm in}\rangle^{\dagger}$$

Let us check that the definition (292) is consistent with the two-point function:

$$\begin{aligned}\langle\Phi_{\rm out}|\Phi_{\rm in}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0}\langle 0|\Phi(z,\bar{z})^{\dagger}\Phi(w,\bar{w})|0\rangle \\ &= \lim_{z,\bar{z},w,\bar{w}\to 0}\bar{z}^{-2h}z^{-2\bar{h}}\langle 0|\Phi(1/\bar{z},1/z)\Phi(w,\bar{w})|0\rangle \\ &= \lim_{\xi,\bar{\xi}\to\infty}\bar{\xi}^{2h}\xi^{2\bar{h}}\langle 0|\Phi(\bar{\xi},\xi)\Phi(0,0)|0\rangle\end{aligned} \tag{298}$$

where $\xi = \frac{1}{z}$ and $\bar{\xi} = \frac{1}{\bar{z}}$.

$$\Phi^{\rm pl}(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right)^{h}\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^{\bar{h}}\Phi^{\rm cyl}(w(z),\bar{w}(\bar{z})) \tag{299}$$

$$z = e^w \tag{300}$$

$$\Phi^{\rm cyl}(w,\bar{w}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \phi_{m,n} e^{-mw} e^{-n\bar{w}} \tag{301}$$

$$\Phi^{\rm pl}(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{302}$$

$$\Phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{303}$$

A straightforward Hermitian conjugation on the real surface yields

$$\Phi(z,\bar{z})^\dagger = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar{z}^{-m-h}\bar{z}^{-n-\bar{h}}\phi^\dagger_{m,n} \tag{304}$$

while the definition (292) implies

$$\begin{aligned} &\Phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\Phi(1/\bar{z},1/z) = \\ &\bar{z}^{-2h}z^{-2\bar{h}}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar{z}^{m+h}z^{n+\bar{h}}\phi_{m,n} = \\ &\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \bar{z}^{-m-h}z^{-n-\bar{h}}\phi_{-m,-n} \end{aligned} \tag{305}$$

Comparing we obtain:

$$\phi^\dagger_{m,n} = \phi_{-m,-n} \tag{306}$$

Applying (292) to the tensor energy-momentum $T(z) = \sum_n z^{-n-2}L_n$ we obtain:

$$T(z)^\dagger = \sum_n \bar{z}^{-n-2}L_n^\dagger \tag{307}$$

and

$$T\left(\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^4} = \sum_n \bar{z}^{n-2}L_n \tag{308}$$

Equating (307) and (308) we receive

$$L_n^\dagger = L_{-n} \tag{309}$$

Other important conditions on the L_n can be derived by requiring the regularity of

$$T(z)|0\rangle = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}|0\rangle \tag{310}$$

at $z=0$. Obviously only terms with $m \leq -2$ are allowed, o we learn that

$$L_m|0\rangle = 0, \qquad m \geq -1 \tag{311}$$

Taking hermitian conjugation we have

$$\langle 0|L_m = 0, \qquad m \leq 1 \tag{312}$$

The only generators annihilating both $|0\rangle$ and $\langle 0|$ are $L_{\pm 1,0}$. This the known already to us the statement of the $SL(2, C)$ invariance of the vacuum.

Now we can derive the condition $c > 0$ in unitary theories

$$\frac{c}{2} = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2 L_2^\dagger|0\rangle \geq 0 \tag{313}$$

since the norm satisfies $||L_2^\dagger|0\rangle||^2 \geq 0$ in a positive Hilbert space.

Let us now consider the state

$$|h, \bar{h}\rangle = \phi(0, 0)|h\rangle \tag{314}$$

created by a holomorphic field $\phi(z)$ of weight h. From the operator product expansion (264) between the energy-momentum tensor T and a primary field we find:

$$[L_n, \phi(w, \bar{w})] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w, \bar{w}) = \tag{315}$$
$$\oint \frac{dz}{2\pi i} z^{n+1} \left(\frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w \phi(w, \bar{w}) \right) = h(n+1)w^n \phi(w, \bar{w}) + w^{n+1}\partial_w \phi(w, \bar{w})$$

so that $[L_n, \phi(0, 0)] = 0$, $n > 0$.

The antiholomorphic counterpart of this relation is

$$[\bar{L}_n, \phi(w, \bar{w})] = \bar{h}(n+1)\bar{w}^n \phi(w, \bar{w}) + \bar{w}^{n+1}\partial_w \phi(w, \bar{w}) \tag{316}$$

Applying this relation to the state (314) we conclude:

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle \qquad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle \tag{317}$$

and

$$L_n|h, \bar{h}\rangle = 0 \qquad \bar{L}_n|h, \bar{h}\rangle = 0 \qquad n > 0 \tag{318}$$

The state satisfying (317) and (318) is known as a highest weight state.

Using (317) and (318) and (281) we can evaluate

$$\langle h|L_{-n}^\dagger L_{-n}|h\rangle = \langle h|[L_n, L_{-n}]|h\rangle = 2n\langle h|L_0|h\rangle + \frac{c}{12}(n^3 - n)\langle h|h\rangle = \tag{319}$$
$$\left(2nh + \frac{c}{12}(n^3 - n)\right)\langle h|h\rangle$$

For $n = 1$ this implies that $h \geq 0$.

Let us consider mode expansion of arbitrary holomorphic field $\phi(z)$ of weight $(h, 0)$

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_m z^{-m-h} \tag{320}$$

$$\phi_m = \frac{1}{2\pi i} \oint dz z^{m+h-1} \phi(z) \tag{321}$$

Regularity of $\phi(z)|0\rangle$ at $z = 0$ requires $\phi_n|0\rangle = 0$ for $n \geq -h+1$. From (314) we see that the state $|h\rangle$ is created by the mode ϕ_{-h}: $|h\rangle = \phi_{-h}|0\rangle$. Now calculate the commutator:

$$[L_n, \phi_m] = \frac{1}{2\pi i} \oint dw w^{m+h-1} (h(n+1) w^n \phi(w) + w^{n+1} \partial_w \phi(w)) \tag{322}$$
$$= \frac{1}{2\pi i} \oint dw w^{m+h+n-1} (h(n+1) - (h+m+n) \phi(w)) =$$
$$(n(h-1) - m) \phi_{m+n}$$

So $[L_0, \phi_m] = -m\phi_m$, implying $L_0|h\rangle = L_0\phi_{-h}|0\rangle = h|h\rangle$.

Lecture 9
Free boson

The action of a free massless boson ϕ is

$$S = \frac{1}{8\pi} \int d^2x \partial_\mu \phi \partial^\mu \phi \tag{323}$$

The equation of motion for the field ϕ is:

$$\Box \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \tag{324}$$

In the polar coordinates we have:

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{325}$$

The solution of the equation or propagator

$$\Box_x G(\mathbf{x}, \mathbf{y}) = \delta^{(2)}(\mathbf{x} - \mathbf{y}) \tag{326}$$

is

$$G(\mathbf{x}, \mathbf{y}) = \log r, \qquad r = (\mathbf{x} - \mathbf{y})^2 \tag{327}$$

or in other words

$$\langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle = -\log(\mathbf{x} - \mathbf{y})^2 \tag{328}$$

in complex coordinates:

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle = -(\log(z - w) + \log(\bar{z} - \bar{w}) \tag{329}$$

Eq. (329) can be derived also remembering that in complex coordinates the propagator satisfies the relation:

$$(\partial_z \partial_{\bar{z}} + \partial_{\bar{z}} \partial_z) G(z, w) = \delta^{(2)}(z - w, \bar{z} - \bar{w}) \tag{330}$$

and

$$\partial_z \frac{1}{\bar{z}} = \partial_{\bar{z}} \frac{1}{z} = \delta^{(2)}(z, \bar{z}) \tag{331}$$

Let us prove (331).

Recall at the beginning the Stokes theorem

$$\int_M d^2z (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) = \int_{\partial M} (A_z dz + A_{\bar{z}} d\bar{z}) \tag{332}$$

Now we can compute:

$$\int_M d^2z\delta(z,\bar{z})f(z) = \int_M d^2z\partial_{\bar{z}}\frac{1}{z}f(z) = \int_M d^2z\partial_{\bar{z}}\frac{f(z)}{z} = \int_{\partial M} dz\frac{f(z)}{z} = f(0) \tag{333}$$

It follows

$$\langle\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle = -\frac{1}{(z-w)^2} \tag{334}$$

$$\langle\partial_{\bar{z}}\phi(z,\bar{z})\partial_{\bar{w}}\phi(w,\bar{w})\rangle = -\frac{1}{(\bar{z}-\bar{w})^2} \tag{335}$$

The energy-momentum tensor of the free boson is

$$T_{\mu\nu} = \frac{1}{4\pi}(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi) \tag{336}$$

Denoting $\partial\phi \equiv \partial_z\phi$ and $\bar{\partial}\phi \equiv \partial_{\bar{z}}\phi$, the holomorphic and anti-holomorphic components of the tensor energy-momentum are

$$T(z) = -\frac{1}{2} : \partial\phi\partial\phi : \tag{337}$$

$$\bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\partial}\phi\bar{\partial}\phi : \tag{338}$$

The normal ordering means:

$$T(z) = -\frac{1}{2}\lim_{w\to z}(\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle) \tag{339}$$

The OPE of $T(z)$ with $\partial\phi$ may be calculated from Wick's theorem:

$$T(z)\partial\phi(w) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \partial\phi(w) \sim \frac{\partial\phi(z)}{(z-w)^2} \tag{340}$$

By expanding $\partial\phi(z)$ around w we arrive at the OPE

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} \tag{341}$$

This shows that $\partial\phi(z)$ is a primary field with conformal dimension 1. Wick's theorem also allows us to calculate the OPE of the energy-momentum tensor with itself:

$$\begin{aligned} T(z)T(w) &= \frac{1}{4} : \partial\phi(z)\partial\phi(z) :: \partial\phi(w)\partial\phi(w) : \\ &\sim \frac{1/2}{(z-w)^4} - \frac{:\partial\phi(z)\partial\phi(w):}{(z-w)^2} \\ &\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned} \tag{342}$$

Let us introduce the vertex operators:

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\phi(z,\bar{z})} : \tag{343}$$

We now demonstrate that these fields are primary with dimensions:

$$h_\alpha = \frac{\alpha^2}{2} \tag{344}$$

We first calculate the OPE of $\partial\phi$ with $\mathcal{V}_\alpha$

$$\partial\phi(z)\mathcal{V}_\alpha(w,\bar{w}) = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\partial\phi(z) : \phi(w,\bar{w}) :^n \tag{345}$$

$$\sim -\frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{(n-1)!} : \phi(w,\bar{w}) :^{n-1}$$

$$\sim -i\alpha\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w}$$

From here we can derive:

$$\left[\frac{1}{2\pi i}\oint_0 i\partial\phi(z)dz, \mathcal{V}_\alpha(w,\bar{w})\right] = \frac{1}{2\pi i}\oint_w \partial\phi(z)\mathcal{V}_\alpha(w,\bar{w})dz = \alpha\mathcal{V}_\alpha(w,\bar{w}) \tag{346}$$

Next we calculate the OPE of $\mathcal{V}_\alpha$ with the energy-momentum tensor:

$$T(z)\mathcal{V}_\alpha(w,\bar{w}) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(i\alpha)^n}{n!} : \partial\phi(z)\partial\phi(z) :: \phi(w,\bar{w}) :^n \tag{347}$$

$$\sim -\frac{1}{2}\frac{1}{(z-w)^2}\sum_{n=2}^{\infty}\frac{(i\alpha)^n}{(n-2)!} : \phi(w,\bar{w}) :^{n-2}$$

$$+\frac{1}{z-w}\sum_{n=1}^{\infty}\frac{(i\alpha)^n}{n!}n : \partial\phi(z) :: \phi(w,\bar{w}) :^{n-1}$$

$$\sim \frac{\alpha^2}{2}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w}$$

To the n-th term in the summation we have applied $2n$ single contractions and $n(n-1)$ double contractions. We have replaced $\partial\phi(z)$ by $\partial\phi(w)$ in th last equation since the difference between the two fields leads to a regular term.

Quantization of the free boson on the cylinder

On a cylinder of circumference L a boson satisfies $\phi(x+L,t) = \phi(x,t)$.

$$\phi(x,t) = \phi_0 + \frac{4\pi a_0 t}{L} + i\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{2\pi n i(x-t)/L} - \bar{a}_n e^{2\pi n i(x+t)/L}\right) \tag{348}$$

From reality of ϕ we have

$$a_n^\dagger = a_{-n} \tag{349}$$

and

$$\bar{a}_n^\dagger = \bar{a}_{-n} \tag{350}$$

Commutation relations follows from the equal-time commutation rules

$$[\phi(x), \phi(x')] = 0, \qquad [\partial_t \phi(x), \partial_t \phi(x')] = 0 \qquad \left[\frac{1}{4\pi}\partial_t \phi(x), \phi(x')\right] = i\delta(x - x') \tag{351}$$

which imply

$$[a_n, a_m] = n\delta_{n+m} \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m} \quad [a_n, \bar{a}_m] = 0 \tag{352}$$

The total momentum of the string is

$$\int_0^L \frac{1}{4\pi}\partial_t \phi(x) = a_0 \tag{353}$$

If we go over to Euclidean space-time (replace t by $-i\tau$) and use the conformal coordinates:

$$z = e^{2\pi(\tau - ix)/L} \qquad \bar{z} = e^{2\pi(\tau + ix)/L} \tag{354}$$

$$\phi(x,t) = \phi_0 - ia_0 \log(z\bar{z}) + i\sum_{n\neq 0} \frac{1}{n}\left(a_n z^{-n} - \bar{a}_n \bar{z}^{-n}\right) \tag{355}$$

$$i\partial\phi(z) = \sum_n a_n z^{-n-1} \tag{356}$$

One has

$$a_n^\dagger = a_{-n} \quad \bar{a}_n^\dagger = \bar{a}_{-n} \tag{357}$$

Commutation relations:

$$[a_n, a_m] = n\delta_{n+m} \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m} \quad [a_n, \bar{a}_m] = 0 \tag{358}$$

can be derived also from the OPE (334)

$$\begin{aligned}
[a_n, a_m] &= i^2\left[\oint \frac{dz}{2\pi i}, \oint \frac{dw}{2\pi i}\right] z^n w^m \partial_z\phi(z)\partial_w\phi(w) \\
&= i^2 \oint \frac{dw}{2\pi i} w^m \oint \frac{dz}{2\pi i} z^n \frac{1}{(z-w)^2} \\
&= \oint \frac{dw}{2\pi i} n w^m w^{n-1} = n\delta_{n+m,0}
\end{aligned} \tag{359}$$

We have that

$$\frac{1}{2\pi i}\oint_0 i\partial\phi(z)dz = a_0 \tag{360}$$

and therefore

$$[a_0,\ \mathcal{V}_\alpha(w,\bar{w})] = \alpha\mathcal{V}_\alpha(w,\bar{w}) \tag{361}$$

Using (337) and remembering the mode expansion of the tensor energy-momentum (276) we obtain:

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m \tag{362}$$

$$L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 \tag{363}$$

let us compute:

$$[L_n, a_k] = \frac{1}{2}\sum_{m\in\mathbb{Z}}[a_{n-m}a_m,\ a_k] = \tag{364}$$
$$\frac{1}{2}\sum_{m\in\mathbb{Z}}(a_{n-m}[a_m,\ a_k] + [a_{n-m},\ a_k]a_m) =$$
$$\frac{1}{2}\sum_{m\in\mathbb{Z}}(a_{n-m}m\delta_{m+k} + a_m(n-m)\delta_{n-m+k}) = \frac{1}{2}(-a_{n+k}k - ka_{n+k}) = -ka_{n+k}$$

We also have

$$[L_0, a_{-m}] = ma_{-m} \qquad [L_0, a_m] = -ma_m \tag{365}$$

$$[L_n,\ L_m] = \frac{1}{2}\sum_{k\in\mathbb{Z}}[L_n,\ a_{m-k}a_k] = \tag{366}$$
$$\frac{1}{2}\sum_{k\in\mathbb{Z}}(a_{m-k}[L_n,\ a_k] + [L_n,\ a_{m-k}]a_k) =$$
$$\frac{1}{2}\sum_{k\in\mathbb{Z}}(-ka_{m-k}a_{n+k} - (m-k)a_{n+m-k}a_k)$$

Now let us bring both terms to the normal ordered form.

For normal ordering we will take the following prescription:

$$: a_i a_j := a_i a_j \quad \text{if} \quad i \leq j \quad \text{and} \quad a_j a_i \quad \text{if} \quad i > j \tag{367}$$

Now for the first term we can write:

$$\sum_{k\in\mathbb{Z}}(-ka_{m-k}a_{n+k}) = \sum_{k\geq\frac{m-n}{2}}(-ka_{m-k}a_{n+k}) + \sum_{k<\frac{m-n}{2}}(-ka_{m-k}a_{n+k}) = \tag{368}$$
$$\sum_{k\geq\frac{m-n}{2}}(-ka_{m-k}a_{n+k}) + \sum_{k<\frac{m-n}{2}}(-ka_{n+k}a_{m-k}) + \sum_{k<\frac{m-n}{2}}(-k)(m-k)\delta_{m+n} =$$
$$\sum_{k\in\mathbb{Z}} :(-ka_{m-k}a_{n+k}): + \sum_{k<m} k(k-m)\delta_{m+n}$$

and for the second

$$\sum_{k\in\mathbb{Z}}(-(m-k)a_{n+m-k}a_k) = \tag{369}$$
$$\sum_{k\geq\frac{m+n}{2}}(-(m-k)a_{n+m-k}a_k) + \sum_{k<\frac{m+n}{2}}(-(m-k)a_{n+m-k}a_k) =$$
$$\sum_{k\geq\frac{m+n}{2}}(-(m-k)a_{n+m-k}a_k) + \sum_{k<\frac{m+n}{2}}(-(m-k)a_k a_{n+m-k}) +$$
$$\sum_{k<\frac{m+n}{2}}(-(m-k)(n+m-k)\delta_{m+n} =$$
$$\sum_{k\in\mathbb{Z}} :(-(m-k)a_{n+m-k}a_k): - \sum_{k<0} k(k-m)\delta_{m+n}$$

Performing in the first term of the last line in (368) the change of the sum variable k to $k' = n+k$ we get

$$\sum_{k'\in\mathbb{Z}} :(-(k'-n)a_{n+m-k'}a_{k'}) := \sum_{k\in\mathbb{Z}} :(-(k-n)a_{n+m-k}a_k): \tag{370}$$

and uniting it the first term in the last line of (369) we obtain:

$$\frac{1}{2}\sum_{k\in\mathbb{Z}} :(-(k-n)a_{n+m-k}a_k): + \frac{1}{2}\sum_{k\in\mathbb{Z}} :(-(m-k)a_{n+m-k}a_k) := (n-m)L_{n+m} \tag{371}$$

For the remaining terms we get:

$$\frac{1}{2}\sum_{k=1}^{m-1} k(k-m)\delta_{m+n} \tag{372}$$

Remembering that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{373}$$

and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{374}$$

we obtain

$$\frac{1}{2}\sum_{k=1}^{m-1} k(k-m)\delta_{m+n} = \frac{1}{2}\delta_{m+n}\Big[\frac{(m-1)m(2m-1)}{6} - m\frac{m(m-1)}{2}\Big] = \tag{375}$$

$$-\frac{1}{12}\delta_{m+n}(m^3-m) = \frac{1}{12}\delta_{m+n}(n^3-n)$$

Now consider the scalar field compactified on circle of radius R:

$$\phi(x+L,t) = \phi(x,t) + 2\pi m R \tag{376}$$

The mode expansion (348) now modifies to

$$\phi(x,t) = \phi_0 + \frac{4\pi n}{RL}t + \frac{2\pi R m}{L}x + i\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{2\pi k i(x-t)/L} - \bar{a}_k e^{2\pi k i(x+t)/L}\right) \tag{377}$$

The center-of-mass momentum is $\frac{L}{4\pi}\partial_t\phi = \frac{n}{R}$, and n should be integer since the vertex operator e^{ipx} should be single valued under the identification $X \equiv X+2\pi R$. If we express this expansion in terms of the complex coordinates z and $\bar{z}$, we find

$$\phi(z,\bar{z}) = \phi_0 - i\left(\frac{n}{R} + \frac{1}{2}Rm\right)\log(z) + i\sum_{k\neq 0}\frac{1}{k}a_k z^{-k} \tag{378}$$

$$-i\left(\frac{n}{R} - \frac{1}{2}Rm\right)\log(\bar{z}) + i\sum_{k\neq 0}\frac{1}{k}\bar{a}_k \bar{z}^{-k}$$

$$i\partial\phi(z) = \left(\frac{n}{R} + \frac{1}{2}Rm\right)\frac{1}{z} + \sum_{k\neq 0} a_k z^{-k-1} \tag{379}$$

The expression (363) for L_0 and $\bar{L}_0$ specialize to

$$L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}\left(\frac{n}{R} + \frac{1}{2}Rm\right)^2 \tag{380}$$

$$\bar{L}_0 = \sum_{n>0} \bar{a}_{-n}\bar{a}_n + \frac{1}{2}\left(\frac{n}{R} - \frac{1}{2}Rm\right)^2 \tag{381}$$

Free Fermion

The action is

$$S = \frac{1}{2\pi}\int d^2x(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) \tag{382}$$

The classical equations of motion are

$$\partial\bar{\psi} = 0 \qquad \text{and} \qquad \bar{\partial}\psi = 0 \tag{383}$$

The propagator is

$$\langle\psi(z)\psi(w)\rangle = \frac{1}{z-w} \tag{384}$$

$$\langle\bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\rangle = \frac{1}{\bar{z}-\bar{w}} \tag{385}$$

$$T^{\bar{z}\bar{z}} = 2\frac{\partial\mathcal{L}}{\partial\bar{\partial}\Phi}\partial\Phi = \frac{1}{\pi}\psi\partial\psi \tag{386}$$

$$T^{zz} = 2\frac{\partial\mathcal{L}}{\partial\partial\Phi}\bar{\partial}\Phi = \frac{1}{\pi}\bar{\psi}\bar{\partial}\bar{\psi} \tag{387}$$

$$T^{z\bar{z}} = 2\frac{\partial\mathcal{L}}{\partial\partial\Phi}\partial\Phi - 2\mathcal{L} = -\frac{1}{\pi}\psi\bar{\partial}\psi \tag{388}$$

$T^{z\bar{z}}$ vanishes on the equation of motion.

The standard holomorphic component is

$$T(z) = -2\pi T_{zz} = \frac{1}{2} : \psi(z)\partial\psi(z) : \tag{389}$$

where, as before we have used the normal-ordered product

$$: \psi(z)\partial\psi(z) := \lim_{w\to z}(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle) \tag{390}$$

$$\begin{aligned} T(z)\psi(w) = -\frac{1}{2} : \psi(z)\partial\psi(z) : \psi(w) &\sim \frac{1}{2}\frac{\partial\psi(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} \\ &\sim \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} \end{aligned} \tag{391}$$

In contracting $\psi(z)$ with $\psi(w)$ we have carried $\psi(w)$ over $\partial\psi(z)$ thus introducing a $(-)$ sign by Pauli's principle. We see from this OPE that the fermion ψ has a conformal dimension $h = \frac{1}{2}$.

The OPE of $T(z)$ with itself is calculated in the same way:

$$\begin{aligned} T(z)T(w) &= \frac{1}{4} : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\ &\sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \end{aligned} \tag{392}$$

Fermion on a plane

On the plane the fermion field has mode expansion:

$$\psi(z) = \sum_k b_k z^{-k-1/2} \tag{393}$$

We find commutation relations:

$$\begin{aligned}\{b_n, b_m\} &= i^2 \left[\oint \frac{dz}{2\pi i}, \oint \frac{dw}{2\pi i} \right] z^{n-1/2} w^{m-1/2} \psi(z)\psi(w) \\ &= i^2 \oint \frac{dw}{2\pi i} w^{m-1/2} \oint \frac{dz}{2\pi i} z^{n-1/2} \frac{-1}{z-w} \\ &= \oint \frac{dw}{2\pi i} w^{m-1/2} w^{n-1/2} = \delta_{n+m,0}\end{aligned} \tag{394}$$

One can have periodic or anti-periodic boundary conditions having half-integer or Neveu-Schwarz (NS) and integer or Ramond modings (R) respectively

$$\begin{aligned}\psi(e^{2\pi i} z) &= \psi(z) \qquad k \in \mathbb{Z} + \frac{1}{2} \quad (NS) \\ \psi(e^{2\pi i} z) &= -\psi(z) \quad k \in \mathbb{Z} \qquad\quad (R)\end{aligned} \tag{395}$$

We calculate first the two-point function in the (NS) sector from the mode expansion

$$\langle \psi(z)\psi(w) \rangle = \sum_{k,q \in \mathbb{Z}+1/2} z^{-k-1/2} w^{-q-1/2} \langle b_k b_q \rangle \tag{396}$$

$$= \sum_{k \in \mathbb{Z}+1/2, k>0} z^{-k-1/2} w^{k-1/2} = \sum_{n=0}^{\infty} \frac{1}{z} \left(\frac{w}{z} \right)^n = \frac{1}{z-w}$$

This agrees with the fermion OPE. However in the (R) the result is different

$$\begin{aligned}\langle \psi(z)\psi(w) \rangle &= \sum_{k,q \in \mathbb{Z}} z^{-k-1/2} w^{-q-1/2} \langle b_k b_q \rangle \\ &= \frac{1}{2\sqrt{zw}} + \sum_{k=1}^{\infty} z^{-k-1/2} w^{k-1/2} \\ &= \frac{1}{\sqrt{zw}} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{w}{z} \right)^k \right]\end{aligned} \tag{397}$$

$$= \frac{1}{2\sqrt{zw}} \frac{z+w}{z-w} = \frac{1}{2} \frac{\sqrt{z/w} + \sqrt{w/z}}{z-w}$$

Now we compute vev of the tensor energy-momentum. We need to use the normal ordering prescription

$$\langle T(z) \rangle = \frac{1}{2} \lim_{\epsilon \to 0} \left(-\langle \psi(z+\epsilon) \partial \psi(z) \rangle + \frac{1}{\epsilon^2} \right) \tag{398}$$

which leads to $\langle T(z)\rangle = 0$ in the (NS) sector, as is trivially verified. In the (R) sector

$$\langle T(z)\rangle = -\frac{1}{4}\mathrm{lim}_{w\to z}\partial_w\left(\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\right)+\frac{1}{2(z-w)^2}=\frac{1}{16z^2} \tag{399}$$

To prove it let us compute the derivative:

$$\partial_w\left(\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\right)=\frac{1}{(z-w)^2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)-\frac{1}{2w^{3/2}z^{1/2}} \tag{400}$$

Now setting $w = z+\epsilon$, where $\epsilon \to 0$ we get

$$\langle T(z)\rangle = -\mathrm{lim}_{\epsilon\to 0}\frac{\left(1+\frac{\epsilon}{z}\right)^{-1/2}+\left(1+\frac{\epsilon}{z}\right)^{1/2}-2}{4\epsilon^2}+\frac{1}{8z^2} \tag{401}$$

Using expansions:

$$\left(1+\frac{\epsilon}{z}\right)^{-1/2}=1-\frac{\epsilon}{2z}+\frac{3}{8}\frac{\epsilon^2}{z^2}+\cdots \tag{402}$$

and

$$\left(1+\frac{\epsilon}{z}\right)^{1/2}=1+\frac{\epsilon}{2z}-\frac{1}{8}\frac{\epsilon^2}{z^2}+\cdots \tag{403}$$

we obtain:

$$\langle T(z)\rangle = \frac{1}{16z^2} \tag{404}$$

Let us introduce the primary field σ with OPE

$$\psi(z)\sigma(w)\sim (z-w)^{1/2}\mu(w)+\cdots \tag{405}$$

and

$$T(z)\sigma(0)|0\rangle \sim \frac{h_\sigma\sigma(0)}{z^2}|0\rangle+\cdots \tag{406}$$

Using this field we can write

$$\langle T(z)\rangle = \langle 0|\sigma(\infty)T(z)\sigma(0)|0\rangle \tag{407}$$

implying $h_\sigma = \frac{1}{16}$.

Fermion Virasoro generators

$$T_{\mathrm{pl}}=\frac{1}{2}\sum_{k,q}\left(k+\frac{1}{2}\right)z^{-q-1/2}z^{-k-3/2}:b_qb_k:= \tag{408}$$

$$\frac{1}{2}\sum_{k,n}\left(k+\frac{1}{2}\right)z^{-n-2}:b_{n-k}b_k:$$

From this, we extract the conformal generator

$$L_n = \frac{1}{2}\sum_k \left(k + \frac{1}{2}\right) : b_{n-k} b_k : \tag{409}$$

If we fix the constant to be added to L_0 from the vacuum energy density we find

$$L_0 = \sum_{k>0} k b_{-k} b_k \quad (\text{NS}) : k \in \mathbb{Z} + 1/2 \tag{410}$$

$$L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \quad (\text{R}) : k \in \mathbb{Z} \tag{411}$$

Fermion vacuum energies on cylinder

Using the formula (274) we obtain

$$(L_0)_{\text{cyl}} = \sum_{k>0} k b_{-k} b_k - \frac{1}{48} \quad (\text{NS}) : k \in \mathbb{Z} + 1/2 \tag{412}$$

$$(L_0)_{\text{cyl}} = \sum_{k>0} k b_{-k} b_k + \frac{1}{24} \quad (\text{R}) : k \in \mathbb{Z} \tag{413}$$

We can obtain the formulas (412) and (413) also in different way, introducing so called ζ-function regularization. We have

$$(L_0)_{\text{cyl}} = \frac{1}{2}\sum_k k : b_{-k} b_k := \sum_{k>0} k b_{-k} b_k - \frac{1}{2}\sum_{k>0} k \tag{414}$$

Boson vacuum energy on cylinder

$$(L_0)_{\text{cyl}} = \frac{1}{2}\sum_n : a_{-n} a_n := \sum_{n>0} a_{-n} a_n + \frac{1}{2}\sum_{n>0} n \tag{415}$$

Therefore we should regularize the sum $\sum_{k>0} k$ for integer and half-integer modings. For this purpose we introduce the ζ-function regularization.

ζ-function regularization

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{416}$$

for Re$z > 1$.

$$\zeta(z) \equiv \zeta(z,1) \tag{417}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{418}$$

Let us change the variable $t = (n+q)y$, $n \in \mathbb{N}$

$$\Gamma(z) = (n+q)^z \int_0^\infty y^{z-1} e^{-(n+q)y} dy \tag{419}$$

$$\frac{\Gamma(z)}{(n+q)^z} = \int_0^\infty y^{z-1} e^{-(n+q)y} dy \tag{420}$$

Let us sum over n remembering (416)

$$\Gamma(z)\zeta(z,q) = \sum_{n=0}^{\infty} \int_0^\infty y^{z-1} e^{-(n+q)y} dy \tag{421}$$

Exchanging order of the sum and the integral we obtain:

$$\Gamma(z)\zeta(z,q) = \int_0^\infty \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy \tag{422}$$

So we arrived to the following integral representation of the ζ- function :

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy \tag{423}$$

Using this expression now we can calculate ζ- function for negative integer values by the analytic continuation . Let us split the integral (423) in two parts:

$$\int_0^\infty \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy = \int_0^1 \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy + \int_1^\infty \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy \tag{424}$$

and replacing

$$\frac{y e^{-qy}}{1-e^{-y}} = \sum_{n=0}^{\infty} (-)^n B_n(q) \frac{y^n}{n!} \tag{425}$$

in the first part. Here $B_n(q)$ is the n-th Bernoulli polynomial. After doing this one can perform the first integration term by term to get

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} (-)^n \frac{B_n(q)}{n!} \frac{1}{z+n-1} + \frac{1}{\Gamma(z)} \int_1^\infty \frac{y^{z-1} e^{-qy}}{1-e^{-y}} dy \tag{426}$$

Remembering that when $z = -m + \epsilon$, $(m \in \mathbb{N})$

$$\Gamma(-m+\epsilon) \sim (-)^m \frac{1}{m!\epsilon} \tag{427}$$

and the remaining integral is finite, the second term vanishes as $\epsilon \to 0$.

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} (-)^n \frac{B_n(q)}{n!} \frac{1}{z+n-1} \tag{428}$$

As for the series, in this limit survives only one term $n = m+1$ yielding the finite limit:

$$\lim_{\epsilon\to 0} \zeta(-m+\epsilon, q) = -\frac{B_{m+1}(q)}{m+1} \tag{429}$$

To evaluate the vacuum energy note that from (416) follows that the infinite sum in (414) can be written as

$$-\frac{1}{2}\zeta(-1,\frac{1}{2}) \tag{430}$$

in NS sector and as

$$-\frac{1}{2}\zeta(-1,1) \tag{431}$$

in R sector.

Note that $B_n(1) \equiv B_n$ is the n-th Bernoulli number defined by the generating function

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} (-)^n B_n \frac{y^n}{n!} \tag{432}$$

To compute necessary for us B_0, B_1, B_2 we rewrite (432) in the form:

$$\frac{1}{y}(y + \frac{y^2}{2} + \frac{y^3}{6} \cdots)(B_0 - B_1 y + B_2 \frac{y^2}{2} \cdots) = \tag{433}$$
$$(1 + \frac{y}{2} + \frac{y^2}{6} \cdots)(B_0 - B_1 y + B_2 \frac{y^2}{2} \cdots) = 1$$

Putting $B_0 = 1$, and equating coefficients of y^n to zero we obtain:

$$\frac{B_0}{2} - B_1 = 0 \tag{434}$$

and

$$\frac{B_2}{2} - \frac{B_1}{2} + \frac{B_0}{6} = 0 \tag{435}$$

From here we have $B_1 = \frac{1}{2}$ and $B_2 = \frac{1}{6}$. Hence we recovered the vacuum energy in the Ramond sector (413). Now we compute the second Bernoulli polynomial.

Using (432) we can rewrite (425) in the form:

$$(B_0 - B_1 y + B_2 \frac{y^2}{2} \cdots)(1 + qy + \frac{q^2 y^2}{2} + \cdots) = (B_0(q) + B_1(q) y + B_2(q) \frac{y^2}{2} \cdots) \tag{436}$$

From here we have

$$B_0(q) = B_0 \tag{437}$$

$$B_1(q) = qB_0 - B_1 \tag{438}$$

and

$$B_2(q) = B_0 q^2 - 2B_1 q + B_2 \tag{439}$$

Hence $B_2(q) = q^2 - q + \frac{1}{6}$, and we find $B_2(\frac{1}{2}) = -\frac{1}{12}$, thus deriving the vacuum energy (412) in the NS sector.

Lecture 10
Correlation functions and OPE of vertex operators

Baker-Campbell-Hausdorff theorem

Let us prove here the Baker-Campbell-Hausdorff theorem

$$e^{A+B} = e^B e^A e^{\frac{C}{2}} \tag{440}$$

where

$$C = [A, B] \tag{441}$$

and satisfies $[C, A] = [C, B] = 0$. Consider the following operator:

$$F(\alpha) = e^{\alpha A + \beta B} \tag{442}$$

$$\begin{aligned}
\frac{dF}{d\alpha} &= \sum_n \frac{1}{n!} \frac{d}{d\alpha}[(\alpha A + \beta B)^n] = \sum_n \frac{1}{n!} \sum_{k=0}^{n-1} (\alpha A + \beta B)^k A (\alpha A + \beta B)^{n-k-1} \\
&= \sum_n \frac{1}{n!} \sum_{k=0}^{n-1} \left((\alpha A + \beta B)^{n-1} A + (n-k-1)(\alpha A + \beta B)^{n-2} \beta C \right) \\
&= \sum_n \frac{1}{n!} \left(n(\alpha A + \beta B)^{n-1} A + \frac{n(n-1)}{2} (\alpha A + \beta B)^{n-2} \beta C \right) \\
&= F(\alpha) \left(A + \frac{\beta C}{2} \right)
\end{aligned} \tag{443}$$

Integrating (443) we obtain:

$$F(\alpha) = F(0) e^{\alpha \left(A + \frac{\beta C}{2} \right)} \tag{444}$$

Remembering that $F(0) = e^{\beta B}$ and C commutes with A we arrive to (440).

Exchanging A and B we also get:

$$e^{A+B} = e^A e^B e^{\frac{-C}{2}} \tag{445}$$

Equating (440) and (445) we derive

$$e^A e^B e^{-\frac{C}{2}} = e^B e^A e^{\frac{C}{2}} \tag{446}$$

and therefore:

$$e^A e^B = e^B e^A e^C \tag{447}$$

Now using (447) we will establish the following identity:

$$\langle : e^{A_1} :: e^{A_2} \cdots : e^{A_n} : \rangle = \exp \sum_{i<j}^{n} \langle A_i A_j \rangle \tag{448}$$

where $A_i = \alpha_i a + \beta_i a^\dagger$ and a and $a^\dagger$ are operators of creation and annihilation satisfying the commutation relation:

$$[a, a^\dagger] = 1 \tag{449}$$

Using (447) we obtain

$$e^{wa} e^{za^\dagger} = e^{za^\dagger} e^{wa} e^{wz} \tag{450}$$

We also have:

$$: e^{A_i} := e^{\beta_i a^\dagger} e^{\alpha_i a} \tag{451}$$

In calculating the normal ordered product of a string $: e^{A_1} :: e^{A_2} \cdots : e^{A_n} :$ of vertex operators we want to bring all the annihilation operators to the right. For instance, it follows from (450) that

$$e^{\alpha_i a} e^{\beta_{i+1} a^\dagger} \cdots e^{\beta_n a^\dagger} = e^{\beta_{i+1} a^\dagger} \cdots e^{\beta_n a^\dagger} e^{\alpha_i a} e^{\alpha_i (\beta_{i+1} + \beta_{i+2} + \cdots \beta_n)} \tag{452}$$

Since $[e^{\alpha_i a}, e^{\alpha_j a}] = 0$, this implies

$$e^{\alpha_i a} : e^{A_{i+1}} : \cdots : e^{A_n} :=: e^{A_{i+1}} : \cdots : e^{A_n} : e^{\alpha_i a} e^{\alpha_i (\beta_{i+1} + \beta_{i+2} + \cdots \beta_n)} \tag{453}$$

Applying this in succession from $i = 1$ to $i = n - 1$, we find

$$: e^{A_1} :: e^{A_2} \cdots : e^{A_n} := e^{(\beta_1 + \cdots + \beta_n) a^\dagger} e^{(\alpha_1 + \cdots + \alpha_n) a} \exp \sum_{i<j}^{n} \alpha_i \beta_j \tag{454}$$

Since $\langle A_i A_j \rangle = \alpha_i \beta_j$ we obtain

$$: e^{A_1} :: e^{A_2} \cdots : e^{A_n} :=: e^{A_1 + \cdots + A_n} : \exp \sum_{i<j}^{n} \langle A_i A_j \rangle \tag{455}$$

Taking expectation value leads to (448).

Since a free field is simply an assembly of decoupled harmonic oscillators, we have

$$: e^{a\phi_1} :: e^{b\phi_2} := e^{a\phi_1 + b\phi_2} : e^{ab\langle \phi_1 \phi_2 \rangle} \tag{456}$$

This relation yields

$$\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta}\mathcal{V}_{\alpha+\beta}(w,\bar{w}) \tag{457}$$

The vertex operator has the form:

$$\mathcal{V}_\alpha(z,\bar{z}) = e^{i\alpha\phi_0} e^{\alpha a_0 \ln(z\bar{z})} V_\alpha^{\rm osc}(z) V_\alpha^{\rm osc}(\bar{z}) \tag{458}$$

where

$$V_\alpha^{\rm osc}(z) =: e^{i\alpha\phi^{\rm osc}(z)} := \exp\Big[-\alpha\sum_{n>0}\frac{1}{n}a_{-n}z^n\Big]\exp\Big[\alpha\sum_{n>0}\frac{1}{n}a_n z^{-n}\Big] \tag{459}$$

where $\phi^{\rm osc}(z)$ is the oscillator part of the scalar, and

$$[\phi_0, a_0] = i \tag{460}$$

From the mode expansion of $\phi^{\rm osc}(z)$

$$\langle\phi^{\rm osc}(z)\phi^{\rm osc}(w)\rangle = -\sum_{n,m}\frac{1}{nm}z^{-n}w^{-m}\langle a_n a_m\rangle = \sum_{n>0}\frac{1}{n}\Big(\frac{w}{z}\Big)^n = -\ln\Big(1-\frac{w}{z}\Big) \tag{461}$$

From here and (455) we have

$$\langle V_{\alpha_1}^{\rm osc}(z_1)\cdots V_{\alpha_n}^{\rm osc}(z_n)\rangle = \prod_{i<j}(z_i-z_j)^{\alpha_i\alpha_j} z_i^{-\alpha_i\alpha_j} \tag{462}$$

For zero modes using

$$[a_0, e^{i\alpha\phi_0}] = \alpha e^{i\alpha\phi_0} \tag{463}$$

we get

$$\langle e^{i\alpha_1\phi_0}e^{\alpha_1 a_0\ln(z_1\bar{z}_1)}\cdots e^{i\alpha_n\phi_0}e^{\alpha_n a_0\ln(z_n\bar{z}_n)}\rangle = \prod_{i<j}|z_i|^{2\alpha_i\alpha_j} \tag{464}$$

when

$$\sum_{i=1}^n \alpha_i = 0 \tag{465}$$

and zero otherwise.

Collecting all we get

$$\langle\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_{-\alpha}(w,\bar{w})\rangle = |z-w|^{-2\alpha^2} \tag{466}$$

$$\langle\mathcal{V}_{\alpha_1}(z_1,\bar{z}_1)\cdots\mathcal{V}_{\alpha_n}(z_n,\bar{z}_n)\rangle = \prod_{i<j}|z_i-z_j|^{4\alpha_i\alpha_j} \tag{467}$$

when

$$\sum_{i=1}^{n} \alpha_i = 0 \tag{468}$$

and zero otherwise.

Let us give also path integral derivation of the vertex operator correlation function. First of all note that the path integral generalization of the integral (5) takes the form:

$$\int \mathcal{D}\phi \exp\left[i\int \phi\Box\phi d^2z + \int id^2zJ(z)\phi(z)\right] = Z\exp\left[\int d^2zd^2z'J(z)G(z,z')J(z')\right] \tag{469}$$

where $G(z,z')$ is free field propagator (327).

According to (52)

$$\langle \exp\left[\int id^2zJ(z)\phi(z)\right]\rangle = \exp\left[\int d^2zd^2z'J(z)G(z,z')J(z')\right] \tag{470}$$

Now correlation function of the vertex operators can be derived taking

$$J(z) = \sum_{i=1}^{N} \alpha_i \delta^2(z - z_i) \tag{471}$$

Inserting (471) in (470) we again obtain (467), omitting because of the normal ordering the coinciding terms $i = j$.

Neutrality via the Ward identity

Let we have a symmetry with the infinitesimal transformation law

$$\Phi'(x) = \Phi(x) - i\omega_a G_a \Phi(x) \tag{472}$$

with the conserved current $j_a^\mu(x)$.

$$\frac{\partial}{\partial x^\mu}\langle j_a^\mu(x)\Phi(x_1)\cdots\Phi(x_n)\rangle \tag{473}$$

$$= -i\sum_{i=1}^{n}\delta(x - x_i)\langle\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_n)\rangle$$

Since the variation of the vertex operator under shift $\phi \to \phi + a$ is $\delta V_\alpha = ia\alpha V$ and the corresponding conserved current is $\jmath^\mu = -\partial^\mu\phi/4\pi$ the relation (473) takes form:

$$\frac{\partial}{\partial x^\mu}\langle\partial^\mu\phi(x)X\rangle = -i\sum_{k=1}^{n}\alpha_k\delta(x - x_k)\langle X\rangle \tag{474}$$

where $X = \mathcal{V}_{\alpha_1}(z_1, \bar{z}_1) \cdots \mathcal{V}_{\alpha_n}(z_n, \bar{z}_n)$.

Integrating the relation (474) over all space we obtain

$$i\langle X\rangle \sum_{k=1}^{n} \alpha_k = \oint dz \langle \partial\phi X\rangle - \oint d\bar{z} \langle \bar{\partial}\phi X\rangle \tag{475}$$

Since the integration contours circle around all space, that is, around point in infinity the integrals hove no singularity outside the contours and the two contour integrals vanish.

Lecture 11
Coulomb gas

OPE

Consider the holomorphic part of the three-point function (288) in the limit $z_1 \to z_2$. The leading term is:

$$\langle 0|\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)|0\rangle = C_{ijk}(z_1 - z_2)^{h_3-h_1-h_2}(z_1 - z_3)^{-2h_3} \tag{476}$$

The last term looks like the propagator of the field ϕ_3 and the expression suggest that the two primary fields ϕ_i and ϕ_j contain in their product the field ϕ_3, with strength C_{ijk}. The precise statement of this fact is the operator product expansion, which says that the product of two operators $O_i(x)$ and $O_j(y)$ in field theory can be expanded in a complete set of operators $O_k(x)$

$$O_i(x)O_j(y) = \sum_k \mathcal{C}_{ijk}(x-y)O_k(x) \tag{477}$$

In conformal field theory we can take as the basis all primaries and a complete set of descendants. Then the operator product expansion has the form:

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\phi_k(w,\bar{w}) + \text{descendants} \tag{478}$$

Action

On a general Riemann surface the action of the scalar field would have a form:

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g}(\partial_\mu\phi\partial^\mu\phi + 2\gamma\phi R) \tag{479}$$

where γ is a constant. The above action is no longer invariant upon a translation $\phi \to \phi + a$. The variation of the action is

$$\delta S = \frac{\gamma a}{4\pi}\int d^2x\sqrt{g}R \tag{480}$$

But the Gauss-Bonnet theorem states that the above expression is a topological invariant:

$$\int d^2x\sqrt{g}R = 8\pi(1-h) \tag{481}$$

where h is the number of handles in the manifold. For the Riemann sphere $h = 0$. Therefore the variation of thr action upon a shift is

$$\delta S = 2a\gamma \tag{482}$$

Since the corresponding Noether current now is not conserved: the Ward identity gets modified:

$$i\sqrt{2}\langle X\rangle \sum_{k=1}^{n} \alpha_k = \oint dz\langle \partial\phi X\rangle - \oint d\bar{z}\langle \bar{\partial}\phi X\rangle + 2\gamma\langle X\rangle \tag{483}$$

Here X stands $X = \mathcal{V}_{\sqrt{2}\alpha_1}(z_1, \bar{z}_1)\cdots\mathcal{V}_{\sqrt{2}\alpha_n}(z_n, \bar{z}_n)$. Denoting $\gamma = i\sqrt{2}\alpha_0$, the neutrality condition (468) modifies to

$$\sum_{i=1}^{n} \alpha_i = 2\alpha_0 \tag{484}$$

Tensor energy-momentum of the boson with the background charge

The tensor-energy momentum

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} - \frac{\gamma}{2\pi}\left(\partial_\mu\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial_\alpha\phi\right) \tag{485}$$

The holomorphic component is then

$$T(z) = -\frac{1}{2} : \partial\phi\partial\phi : + i\sqrt{2}\alpha_0\partial^2\phi \tag{486}$$

We calculate the OPE of the energy-momentum tensor with the primary field of the free boson and with itself. We have to look only at the extra term $i\sqrt{2}\alpha_0\partial^2\phi$. We easily find:

$$T(z)\partial\phi(w) = \frac{2\sqrt{2}i\alpha_0}{(z-w)^3} + \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} \tag{487}$$

We see that $\partial\phi(w)$ is no longer a primary field. However the vertex operators are still primary:

$$\partial^2\phi(w)V_\alpha(w) \sim \frac{i\sqrt{2}\alpha}{(z-w)^2}V_\alpha(w) \tag{488}$$

what means that the conformal dimension of V_α is now

$$h_\alpha = \alpha^2 - 2\alpha_0\alpha \tag{489}$$

The dimension (489) is invariant under $\alpha \to 2\alpha_0 - \alpha$. Therefore the vertex operators V_α and $V_{2\alpha_0-\alpha}$ have the same dimension.

The OPE of T with itself receives the following contribution from the extra term:

$$\begin{aligned} -\frac{1}{2}i\sqrt{2}\alpha_0 : \partial\phi(z)\partial\phi(z) : \partial^2\phi(w) &= i\sqrt{2}\alpha_0\partial_w\left[\frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{(z-w)} + \text{reg}\right] \quad (490) \\ &= i\sqrt{2}\alpha_0\left[\frac{2\partial\phi(w)}{(z-w)^3} + \frac{2\partial^2\phi(w)}{(z-w)^2} + \frac{\partial^3\phi(w)}{(z-w)} + \text{reg}\right] \end{aligned}$$

$$-\frac{1}{2}i\sqrt{2}\alpha_0\partial^2\phi(z) : \partial\phi(w)\partial\phi(w) : \tag{491}$$

$$= -i\sqrt{2}\alpha_0\partial_z\left[-\frac{\partial\phi(w)}{(z-w)^2}\right] = -i2\sqrt{2}\alpha_0\frac{\partial\phi(w)}{(z-w)^3}$$

$$-2\alpha_0^2\partial^2\phi(z)\partial^2\phi(w) = \frac{-12\alpha_0^2}{(z-w)^4} \tag{492}$$

Summing all these contributions we find usual OPE of T with itself with the central charge

$$c = 1 - 24\alpha_0^2 \tag{493}$$

Consider the operator ψ of the conformal dimension $h_\psi = 1$. Its integral

$$A = \oint dz\psi(z) \tag{494}$$

is invariant under conformal transformation.

There are only two local fields of dimension 1 available for the construction of screening operators: the vertex operators $V_\pm$ defined as:

$$V_\pm \equiv V_{\alpha_\pm} \tag{495}$$

where

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{496}$$

One can check that the conformal dimension is

$$\alpha_\pm^2 - 2\alpha_\pm\alpha_0 = 1 \tag{497}$$

Note that

$$\alpha_+ + \alpha_- = 2\alpha_0 \tag{498}$$

$$\alpha_+\alpha_- = -1 \tag{499}$$

Hence we have two screening operators

$$Q_\pm = \oint dzV_\pm(z) = \oint dze^{i\sqrt{2}\alpha_\pm\phi(z)} \tag{500}$$

Inserting Q_+ or Q_- an integer number of times in a correlator will not affect its conformal properties but will completely screen the charge in some cases, since Q_+ and Q_- carry charges α_+ and α_- respectively. The modified two-point function

$$\langle V_\alpha(z)V_\alpha(w)Q_+^mQ_-^n\rangle \tag{501}$$

should satisfy the neutrality condition

$$2\alpha + m\alpha_+ + n\alpha_- = 2\alpha_0 = \alpha_+ + \alpha_- \tag{502}$$

Accordingly the admissible charges are

$$\alpha_{m,n} = \frac{1}{2}(1-m)\alpha_+ + \frac{1}{2}(1-n)\alpha_- \tag{503}$$

and denote

$$V_{m,n} = V_{\alpha_{m,n}} \tag{504}$$

The conformal dimensions of these fields are

$$h_{m,n}(c) = \frac{1}{4}(m\alpha_+ + n\alpha_-)^2 - \alpha_0^2 \tag{505}$$

Lecture 12
Minimal models

Consider α_+ and α_- satisfying the relation

$$p'\alpha_+ + p\alpha_- = 0 \tag{506}$$

for some integers p and p' $(p > p')$. Then we have the periodicity condition

$$\alpha_{r+p',s+p} = \alpha_{r,s} \tag{507}$$

Using (498) and (499) we obtain:

$$\alpha_+ = \sqrt{p/p'} \tag{508}$$

and

$$\alpha_- = -\sqrt{p'/p} \tag{509}$$

from which it follows that

$$\alpha_{m,n} = \frac{1}{2\sqrt{p'p}}\Big[p(1-m) - p'(1-n)\Big] \qquad \alpha_0 = \frac{p-p'}{2\sqrt{p'p}} \tag{510}$$

The relation leads:

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{511}$$

$$h_{m,n} = \frac{(mp-np')^2 - (p-p')^2}{4pp'} \tag{512}$$

These are famous relations for minimal models.

The conformal dimensions (512) satisfy

$$h_{m,n} = h_{p'+m,p+n} \tag{513}$$

$$h_{m,n} = h_{p'-m,p-n} \tag{514}$$

Let us derive the fusion rules. We want to find the fields $\phi_{k,l}$ appearing in the OPE of $\phi_{m,n}$ and $\phi_{r,s}$. To do this we only need to concentrate on the thee-point function. In the Coulomb gas representations, and using $SL(2,C)$ invariance, there are three equivalent ways of representing the three point function:

$$\langle V_{\overline{k,l}}(\infty)V_{m,n}(1)V_{r,s}(0)Q_+^{t_+}Q_-^{t_-}\rangle \tag{515}$$

$$\langle V_{k,l}(\infty)V_{\overline{m,n}}(1)V_{r,s}(0)Q_+^{t'_+}Q_-^{t'_-}\rangle \tag{516}$$

$$\langle V_{k,l}(\infty)V_{m,n}(1)V_{\overline{r,s}}(0)Q_+^{t''_+}Q_-^{t''_-}\rangle \tag{517}$$

The notation $V_{\overline{\alpha}}$ indicates here out-state $V_{2\alpha_0-\alpha}$. The neutrality condition (484) of (515) implies:

$$2\alpha_0 - \alpha_{k,l} + \alpha_{m,n} + \alpha_{r,s} + t_+\alpha_+ + t_-\alpha_- = 2\alpha_0 \tag{518}$$

or

$$-\frac{1}{2}(1-k)\alpha_+ - \frac{1}{2}(1-l)\alpha_- + \frac{1}{2}(1-m)\alpha_+ + \frac{1}{2}(1-n)\alpha_- \\ +\frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- + t_+\alpha_+ + t_-\alpha_- = 0 \tag{519}$$

implying

$$k - m + 1 - r + 2t_+ = 0 \tag{520}$$

and

$$l - n + 1 - s + 2t_- = 0 \tag{521}$$

From here we obtain

$$k \leq m + r - 1 \qquad \text{and} \qquad m + r - k - 1 \;\text{ is even} \tag{522}$$

$$l \leq n + s - 1 \qquad \text{and} \qquad n + s - l - 1 \;\text{ is even} \tag{523}$$

Similar constraints come (516) and (517) thus leading to

$$\begin{aligned} k &\leq m + r - 1 \\ m &\leq k + r - 1 \\ r &\leq m + k - 1 \\ m + r + k &\;\text{ is odd} \end{aligned} \tag{524}$$

and

$$\begin{aligned} l &\leq n + s - 1 \\ n &\leq s + l - 1 \\ s &\leq l + n - 1 \\ l + n + s &\;\text{ is odd} \end{aligned} \tag{525}$$

Conditions (524) and (525) imply

$$\phi_{m,n} \times \phi_{r,s} = \sum_{k=|m-r|+1, k+m+r=\text{odd}}^{m+r-1} \sum_{l=|n-s|+1, n+l+s=\text{odd}}^{n+s-1} \phi_{k,l} \tag{526}$$

Eq. (514) requires that simultaneously

$$\phi_{m,n} \times \phi_{r,s} = \sum_{k=|m-r|+1, k+m+r=\text{odd}}^{2p'-m-r-1} \sum_{l=|n-s|+1, n+l+s=\text{odd}}^{2p-n-s-1} \phi_{k,l} \tag{527}$$

Eqs. (526) and (527) are compatible as long as

$$\phi_{m,n} \times \phi_{r,s} = \sum_{k=|m-r|+1, k+m+r=\text{odd}}^{\min(m+r-1, 2p'-m-r-1)} \sum_{l=|n-s|+1, n+l+s=\text{odd}}^{\min(n+s+1, 2p-n-s-1)} \phi_{k,l} \tag{528}$$

what is well known fusion rule for minimal models.

It is then straightforward matter to see that the following set of indices

$$1 \leq r < p' \qquad 1 \leq s < p \tag{529}$$

closes under the above formula. This therefore constitute a legitimate truncation of the set of admissible charges $\alpha_{r,s}$ in the sense that the operator algebra closes within this set.

Unitary minimal model

If one requires that $h_{m,n}$ as defined in (512) is always not negative one obtains the condition of unitary minimal models

$$|p' - p| = 1 \tag{530}$$

Ising model

Take $p' = 3$ and $p = 4$. Then $c = \frac{1}{2}$, $\alpha_+ = \frac{2}{\sqrt{3}}$, and $\alpha_- = -\frac{\sqrt{3}}{2}$.

$$\alpha_{2,1} = -\frac{\alpha_+}{2} \qquad h_{2,1} = \frac{1}{2} \qquad \epsilon = e^{i\sqrt{2}\alpha_{2,1}\phi} \tag{531}$$

$$\alpha_{1,2} = -\frac{\alpha_-}{2} \qquad h_{1,2} = \frac{1}{16} \qquad \sigma = e^{i\sqrt{2}\alpha_{1,2}\phi} \tag{532}$$

The corresponding OPE is

$$\begin{aligned} \sigma \times \sigma &= \mathbb{I} + \epsilon \\ \sigma \times \epsilon &= \sigma \\ \epsilon \times \epsilon &= \mathbb{I} \end{aligned} \tag{533}$$

Lecture 13
CFT on torus and Modular transformation

Torus

A torus may be defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. On the complex plane these lattice vectors may be represented by two complex numbers ω_1 and ω_2 which we call the periods of the lattice and hence we have

$$w \approx w + n\omega_1 + m\omega_2 \tag{534}$$

Naturally the properties of conformal field theories defined on a torus do not depend on the overall scale of the lattice, nor on the absolute orientation of the lattice vectors. The relevant parameter is the ration $\tau = \omega_2/\omega_1$, the so called modular parameter. Hence we can choose $\omega_2 = 2\pi\tau$ and $\omega_1 = 2\pi$.

Partition function on torus

Conformal field theory on a cylinder coordinatized by w can now be transfered to a torus as follows. We let H and P denote the energy and momentum operators, i.e. the operators that effect translations in the space and time directions Rew and Imw respectively. On the plane we saw that $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ respectively generated dilatations and rotations, so according to the discussion of radial quantization we have $H = (L_0)_{\rm cyl} + (\bar{L}_0)_{\rm cyl}$ and $P = (L_0)_{\rm cyl} - (\bar{L}_0)_{\rm cyl}$. To define a torus we need to identify two periods in w. It is convenient to redefine $w \to iw$ and as we discussed before to choose $w \equiv w + 2\pi$ and $w \equiv w + 2\pi\tau$. Denote by τ_1 and τ_2 real and imaginary parts of τ

$$\tau = \tau_1 + i\tau_2 \tag{535}$$

This means that the surfaces Im$w = 2\pi\tau_2$ and Im$w = 0$ are identified after a shift by Re$w \to$ Re$w + 2\pi\tau_1$. Since we are defining (imaginary) time translation of Imw by its period $2\pi\tau_2$ to be accompanied by a spatial translation of Rew by $2\pi\tau_1$, the operator implementation for the partition function of a theory on torus with modular parameter τ is

$$\begin{aligned} Z = \int e^{-S} &= {\rm Tr} e^{2\pi i\tau_1 P} e^{-2\pi\tau_2 H} = {\rm Tr} e^{2\pi i\tau_1\left((L_0)_{\rm cyl} - (\bar{L}_0)_{\rm cyl}\right)} e^{-2\pi\tau_2\left((L_0)_{\rm cyl} + (\bar{L}_0)_{\rm cyl}\right)} \\ &= {\rm Tr} e^{2\pi i\tau (L_0)_{\rm cyl}} e^{-2\pi i\bar{\tau}(\bar{L}_0)_{\rm cyl}} = {\rm Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} = (q\bar{q})^{-\frac{c}{24}} {\rm Tr} q^{L_0} \bar{q}^{\bar{L}_0} \end{aligned} \tag{536}$$

where $q = e^{2\pi i\tau}$.

Modular Invariance

The main advantage of studying conformal field theories on a torus is the imposition of constraints on the operator content of the theory from the requirement that the partition function be independent of the choice of periods ω_1 and ω_2 for a given torus.

We let ω_1' and ω_2' be two periods describing the same lattice as ω_1 and ω_2. Since the points ω_1' and ω_2' belongs to the lattice, they must be expressible as integer combinations of ω_1 and ω_2:

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{aligned} \tag{537}$$

where $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$.

These transformations (537) form group $SL(2, \mathbb{Z})$.

Under the change of period (537) the modular parameter transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{538}$$

The generators of the transformations (538) are

$$T : \tau \to \tau + 1 \tag{539}$$

and

$$S : \tau \to -\frac{1}{\tau} \tag{540}$$

The Hilbert space of the conformal field theory has the form:

$$\mathcal{H} = \oplus_{i,\bar{i}} R_i(c) \otimes R_{\bar{i}}(c) \tag{541}$$

$R_i(c)$ is the chiral algebra highest weight i representation. Hence defining the character

$$\chi_i(\tau) = \mathrm{Tr}_{R_i} q^{L_0 - c/24} \tag{542}$$

one can write

$$Z(\tau) = \sum_{i,\bar{h}} N_{i,\bar{i}} \chi_i(\tau) \bar{\chi}_{\bar{i}}(\bar{\tau}) \tag{543}$$

where $N_{i,\bar{i}}$ denotes multiplicity of occurrence of $R_i(c) \otimes R_{\bar{i}}(c)$ in $\mathcal{H}$. The first obvious condition for the partition function to be modular invariant is that the characters $\chi_i(\tau)$ define a representation space of the modular transpositions:

$$S : \chi_i \to S_i^j \chi_j \tag{544}$$

$$T : \chi_i \to e^{2\pi i(h_i - c/24)}\chi_i \tag{545}$$

where h_i is the conformal weight of the highest weight i. The matrix $N_{i,\bar{i}}$ in the partition function is determined by demanding modular invariance of the partition function of the model.

The free fermion torus partition function

To compute partition function of the fermion on torus we should specify boundary condition of fermion in both direction. The fermion periodic (P) in space direction has integer moding, and fermion antiperiodic (A) in space direction has half-integer moding. Also switching the boundary condition from the anti-periodic to the periodic in the time direction is reached by the inserting of the parity operator, $(-)^F$ anticommuting with the fermion field ψ , where F is fermion number operator. Denoting by the first index the boundary condition in the space direction, and by the second the boundary condition in the time direction and also remembering vacuum energies we obtain:

$$Z_{P,P} = \frac{1}{\sqrt{2}}\mathrm{Tr}(-)^F q^{L_0 - 1/48} = \frac{1}{\sqrt{2}}\mathrm{Tr}(-)^F q^{\sum_k kb_{-k}b_k + 1/24} \tag{546}$$

$$Z_{P,A} = \frac{1}{\sqrt{2}}\mathrm{Tr}q^{L_0 - 1/48} = \frac{1}{\sqrt{2}}\mathrm{Tr}q^{\sum_k kb_{-k}b_k + 1/24} \tag{547}$$

$$Z_{A,P} = \mathrm{Tr}(-)^F q^{L_0 - 1/48} = \mathrm{Tr}(-)^F q^{\sum_k kb_{-k}b_k - 1/48} \tag{548}$$

$$Z_{A,A} = \mathrm{Tr}q^{L_0 - 1/48} = \mathrm{Tr}q^{\sum_k kb_{-k}b_k - 1/48} \tag{549}$$

These partition functions may be easily calculated, since q^{L_0} factorizes into an infinite product of operators, one for each fermion mode (the same is true of $(-)^F$. For example

$$Z_{A,P} = q^{-1/48}\mathrm{Tr}\prod_{k>0} q^{kb_{-k}b_k}(-)^{F_k} = q^{-1/48}\prod_{k>0}\left(\mathrm{Tr}q^{kb_{-k}b_k}(-)^{F_k}\right) \tag{550}$$

For a given fermion mode, there are only two states and the traces are trivially calculated:

$$\mathrm{Tr}q^{kb_{-k}b_k} = 1 + q^k \tag{551}$$

$$\mathrm{Tr}q^{kb_{-k}b_k}(-)^{F_k} = 1 - q^k \tag{552}$$

We may therefore write the following infinite product for the partition functions, and relate them to the theta functions:

$$Z_{P,P} = \frac{1}{\sqrt{2}} q^{1/24} \prod_{n=0}^{\infty} (1 - q^n) = 0 \tag{553}$$

$$Z_{P,A} = \frac{1}{\sqrt{2}} q^{1/24} \prod_{n=0}^{\infty} (1 + q^n) = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} \tag{554}$$

$$Z_{A,P} = q^{-1/48} \prod_{r=1/2}^{\infty} (1 - q^r) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \tag{555}$$

$$Z_{A,A} = q^{-1/48} \prod_{r=1/2}^{\infty} (1 + q^r) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \tag{556}$$

$$\theta_2(\tau) = 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2 \tag{557}$$

$$\theta_3(\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2})^2 \tag{558}$$

$$\theta_4(\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2})^2 \tag{559}$$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{560}$$

These functions have following modular transformation properties:

$$\begin{aligned} \theta_2(-1/\tau) &= \sqrt{-i\tau}\theta_4(\tau) \\ \theta_3(-1/\tau) &= \sqrt{-i\tau}\theta_3(\tau) \\ \theta_4(-1/\tau) &= \sqrt{-i\tau}\theta_2(\tau) \\ \eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) \end{aligned} \tag{561}$$

$$\begin{aligned} \theta_2(\tau+1) &= e^{i\pi/4}\theta_2(\tau) \\ \theta_3(\tau+1) &= \theta_4(\tau) \\ \theta_4(\tau+1) &= \theta_3(\tau) \\ \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \end{aligned} \tag{562}$$

The modular invariant partition functions has the form:

$$Z = |Z_{P,A}|^2 + |Z_{A,P}|^2 + |Z_{A,A}|^2 \tag{563}$$

The Virasoro characters are

$$\chi_{1,1} = \frac{1}{2}(Z_{A,A} + Z_{A,P}) \tag{564}$$

$$\chi_{2,1} = \frac{1}{2}(Z_{A,A} - Z_{A,P}) \tag{565}$$

$$\chi_{1,2} = \frac{1}{\sqrt{2}} Z_{P,A} \tag{566}$$

In the terms of Virasoro characters the partition function (563) takes form

$$Z = 2(|\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{1,2}|^2) \tag{567}$$

The matrix of the modular transformation is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{568}$$

Lecture 14
Free boson on torus

Recall the action of the free boson:

$$S = \frac{1}{2\pi} \int \partial\phi \bar{\partial}\phi \tag{569}$$

We assume a bosonic coordinate compactified on a circle of radius R:

$$\phi \equiv \phi + 2\pi R \tag{570}$$

That means that there exist instanton sectors with n, n' windings of the boson on a torus:

$$\phi_0(z+\tau, \bar{z}+\bar{\tau}) = \phi_0(z,\bar{z}) + 2\pi R n' \tag{571}$$

$$\phi_0(z+1, \bar{z}+1) = \phi_0(z,\bar{z}) + 2\pi R n \tag{572}$$

A doublet of integers (n, n') then specifies a topological class of configurations obeying the above periodicity conditions, and a partition function $Z_{n,n'}$ is defined by integrating over the configurations of such a class. The integration may be done by decomposing over the configurations of such a class. The integration may be done by decomposing ϕ into a special configuration, which is also a classical solution to the equation of motion , $\phi_0^{n,n'}$ (with vanishing Laplacian) and a periodic field. This reads

$$\phi = \phi_0^{n,n'} + \tilde{\phi} \tag{573}$$

$$\phi_0^{n,n'} = 2\pi R \frac{1}{2i\tau_2}(n'(z-\bar{z}) + n(\tau\bar{z} - \bar{\tau}z) \tag{574}$$

The action $S(\phi)$ is then the sum of $S(\tilde{\phi})$ (the action of the periodic field) plus the action $S(\phi_0^{n,n'})$ of the classical linear configuration. Indeed, since $\Box\phi_0^{n,n'} = 0$, the crossed terms in the action $S(\phi)$ are proportional to

$$\int d^2x \partial_\mu \phi_0^{n,n'} \partial^\mu \tilde{\phi} = -\int d^2x \tilde{\phi} \Box \phi_0^{n,n'} = 0 \tag{575}$$

where an integration by parts has been performed. $S(\phi_0^{n,n'})$ is easily calculated

$$S(\phi_0^{n,n'}) = \frac{1}{2\pi}\int \partial\phi_0^{n,n'} \bar{\partial}\phi_0^{n,n'} = \frac{\pi R^2}{2\tau_2}|n' - n\tau|^2 \tag{576}$$

Here we have taken into account that the torus area $A = \tau_2$. Hence the path integral can be written as

$$\int \mathcal{D}\phi e^{-S} = \sum_{n,n'=-\infty}^{\infty} e^{-S(\phi_0^{n,n'})} \int \mathcal{D}'\tilde{\phi} e^{-S(\tilde{\phi})} \tag{577}$$

where the prime in the integration measure $\mathcal{D}'\tilde{\phi}$ indicates that the constant part is excluded. Now remembering the values of the gaussian integrals we can formally write

$$\int \mathcal{D}'\tilde{\phi} e^{-S(\tilde{\phi})} = \int \mathcal{D}'\tilde{\phi} e^{\int d^2x \tilde{\phi}\Box\tilde{\phi}} = \frac{\sqrt{2}R}{\sqrt{\det'\Box}}\sqrt{\tau_2} \tag{578}$$

To compute (578) we will expand the field ϕ along the normalized eigenfunctions ϕ_n of the $\Box$ with eigenvalues $-\lambda_n$:

$$\phi(x) = \sum_n c_n \phi_n \tag{579}$$

The functional integral over the nonzero modes is then

$$\int \mathcal{D}'\tilde{\phi} e^{-S(\tilde{\phi})} = \int \prod_i \frac{dc_i}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}\sum_n \lambda_n c_n^2\Big) = \prod_n \Big(\frac{1}{\lambda_n}\Big)^{1/2} \tag{580}$$

The additional factor $R\sqrt{2\tau_2}$ comes from the constant mode integration. The factor $\sqrt{\tau_2} = \sqrt{A}$ comes from the normalization of the zero mode, and factor R comes from the integration of the constant part :$\int d\phi_0 = 2\pi R$.

To evaluate $\det'\Box$ as a formal product of eigenvalues, we work with a basis of eigenfunctions:

$$\psi_{nm} = e^{2\pi i \frac{1}{2i\tau_2}(n(z-\bar{z})+m(\tau\bar{z}-\bar{\tau}z))} \tag{581}$$

single-valued under both $z \to z+1$ and $z \to z+\tau$. The regularized determinant is defined by omitting the eigenfunction with $n = m = 0$

$$\det'\Box = \prod_{m,n\neq 0,0} \frac{\pi^2}{\tau_2^2}(n-\tau m)(n-\bar{\tau}m) \tag{582}$$

The infinite product may be evaluated using again ζ-function regularization. Recall the following special values of ζ-function (416):

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{583}$$

$$\zeta(-1) = -\frac{1}{12}, \qquad \zeta(0) = -\frac{1}{2} \tag{584}$$

and

$$\zeta'(0) = -\frac{1}{2}\ln 2\pi \tag{585}$$

In this regularization scheme we have:

$$\prod_{n=1}^{\infty} a = a^{\zeta(0)} = a^{-1/2} \tag{586}$$

and

$$\prod_{-\infty}^{\infty} a = a^{2\zeta(0)+1} = 1 \tag{587}$$

$$\prod_{m=1}^{\infty} q^{-m} = q^{-\zeta(-1)} = q^{1/12} \tag{588}$$

Taking into account that

$$\zeta'(0) = -\sum_{n=1}^{\infty} \ln n \tag{589}$$

we obtain:

$$\prod_{n=1}^{\infty} n^{\alpha} = e^{-\alpha\zeta'(0)} = (2\pi)^{\alpha/2} \tag{590}$$

From (587) we get

$$\prod_{m,n\neq 0,0} \frac{\pi^2}{\tau_2^2} = \frac{\tau_2^2}{\pi^2}\prod_{m,n}\frac{\pi^2}{\tau_2^2} = \frac{\tau_2^2}{\pi^2} \tag{591}$$

From (590) we derive

$$\prod_{n\neq 0} n^2 = (2\pi)^2 \tag{592}$$

Now using (591) we can write:

$$\det{}'\Box = \prod_{m,n\neq 0,0}\frac{\pi^2}{\tau_2^2}(n-\tau m)(n-\bar{\tau}m) = \frac{\tau_2^2}{\pi^2}\prod_{m,n\neq 0,0}(n-\tau m)(n-\bar{\tau}m) \tag{593}$$

Then separating $m = 0$ and $m \neq 0$ terms and using (592) we derive

$$\det{}'\Box = \frac{\tau_2^2}{\pi^2}\prod_{n\neq 0} n^2 \prod_{m\neq 0, n\in\mathbb{Z}}(n-\tau m)(n-\bar{\tau}m) = \frac{\tau_2^2}{\pi^2}(2\pi)^2\prod_{m\neq 0, n\in\mathbb{Z}}(n-\tau m)(n-\bar{\tau}m) \tag{594}$$

Using

$$\pi a\prod_{n=1}^{\infty}\left(1-\frac{a^2}{n^2}\right) = \sin\pi a \tag{595}$$

and (592), (586) we can also establish:

$$\prod_{n=-\infty}^{\infty}(n+a) = a\prod_{n=1}^{\infty}(-n^2)\left(1-\frac{a^2}{n^2}\right) = 2i\sin\pi a \tag{596}$$

Now separating $m > 0$ and $m < 0$ terms and using (596) we obtain

$$\begin{aligned}\det{}'\Box = \frac{\tau_2^2}{\pi^2}(2\pi)^2 \prod_{m>0,n\in\mathbb{Z}} (n-\tau m)(n+\tau m)(n-\bar{\tau}m)(n+\bar{\tau}m) \\ = 4\tau_2^2\prod_{m>0}(e^{-\pi i m\tau}-e^{\pi i m\tau})^2(e^{-\pi i m\bar{\tau}}-e^{\pi i m\bar{\tau}})^2 \\ = 4\tau_2^2\prod_{m>0}(q\bar{q})^{-m}(1-q^m)^2(1-\bar{q}^m)^2\end{aligned} \tag{597}$$

And remembering (588) we end up with

$$\det{}'\Box = 4\tau_2^2(q\bar{q})^{1/12}\prod_{m>0}(1-q^m)^2(1-\bar{q}^m)^2 = 4\tau_2^2\eta^2\bar{\eta}^2 \tag{598}$$

Inserting (598) in (578) we obtain the first contribution to the partition function:

$$\int \mathcal{D}'\tilde{\phi}e^{\int d^2x\tilde{\phi}\Box\tilde{\phi}} = \frac{R}{\sqrt{2\tau_2}}\frac{1}{\eta\bar{\eta}} \tag{599}$$

Note that the expression (599) is modular invariant. Under the modular transformation S

$$\tau_2 \to \frac{\tau_2}{|\tau|^2} \tag{600}$$

Remembering the modular transformation of the η function

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau) \tag{601}$$

we see that (599) is indeed modular invariant.

Now let us turn to the instanton contribution:

$$Z_{\text{inst}} = \sum_{n,n'=-\infty}^{\infty} e^{-S(\phi_0^{n,n'})} = \sum_{n,n'=-\infty}^{\infty} e^{-\frac{\pi R^2}{2\tau_2}|n'-n\tau|^2} \tag{602}$$

It is simple to check that (602) is modular invariant as well. Under a general $SL(2,\mathbb{Z})$ mapping $\tau \to \frac{a\tau+b}{c\tau+d}$ the τ dependent part of the exponent becomes

$$\frac{|n'-n\tau|^2}{\text{Im}\tau} \to \frac{|n'-(na\tau+nb)/(c\tau+d)|^2|c\tau+d|^2}{\text{Im}[(a\tau+b)(c\bar{\tau}+d)]} \tag{603}$$

Using that $ad - bc = 1$ we easily obtain:

$$\mathrm{Im}[(a\tau + b)(c\bar{\tau} + d)] = \mathrm{Im}\tau \tag{604}$$

hence

$$\frac{|n' - n\tau|^2}{\mathrm{Im}\tau} \to \frac{|n'c\tau + n'd - (na\tau + nb)|^2}{\mathrm{Im}\tau} \tag{605}$$

Thus the modular transformation acts on n, n' doublet as the following $SL(2, \mathbb{Z})$ mapping :

$$n \to na - n'c \tag{606}$$

$$n' \to -nb + n'd \tag{607}$$

Since the $SL(2, \mathbb{Z})$ mapping does not change the lattice of doublets n, n' the sum (602) is modular invariant. For example under the generators T and S the doublets of windings transform:

$$T: \qquad n \to n \qquad \text{and} \qquad n' \to -n + n' \tag{608}$$

and

$$S: \qquad n \to n' \qquad \text{and} \qquad n' \to -n \tag{609}$$

obviously leaving the full sum over n, n' invariant.

For the purpose to compare the partition function (577) computed in the path integral approach, with the partition function calculated in the operator approach, we rewrite now the instanton contribution in the different for, which can be reached via Poisson resummation. Let us at the beginning recall the Poisson resummation formula.

Poisson resummation formula

$$\sum_m f(m) = \sum_n \tilde{f}(n) \tag{610}$$

where $\tilde{f}(p)$ is the Fourier transform of $f(x)$

$$\tilde{f}(p) = \int_{-\infty}^{\infty} e^{2\pi i x p} f(x) dx \tag{611}$$

To prove this relation we introduce the auxiliary function

$$F(z) = \sum_{n=-\infty}^{\infty} f(z + n) \tag{612}$$

This function is manifestly a periodic function of z, which can thus be Fourier expanded as

$$F(z) = \sum_{p=-\infty}^{\infty} e^{-2\pi i z m}\tilde{F}(m) \tag{613}$$

with

$$\tilde{F}(m) = \int_0^1 dy e^{2\pi i y m} F(y) \tag{614}$$

Now we substitute (614) in (613) and also use the definition of $F(z)$ to obtain:

$$F(z) = \sum_{m=-\infty}^{\infty} e^{-2\pi i z m} \int_0^1 dy e^{2\pi i y m} \sum_{n=-\infty}^{\infty} f(y+n) \tag{615}$$

Using that $e^{2\pi i y m} = e^{2\pi i (y+n) m}$ we can change the variable $y' = y+n$. Combined summation over n and integration by y can be written as an integration over the whole $\mathbb{R}$, thus yielding

$$F(z) = \sum_{m=-\infty}^{\infty} e^{-2\pi i z m} \int_{-\infty}^{\infty} e^{2\pi i y' m} f(y') dy' \tag{616}$$

Using (611) we obtain:

$$F(z) = \sum_{m=-\infty}^{\infty} e^{-2\pi i z m} \tilde{f}(m) \tag{617}$$

The Poisson resummation formula (610) can be derived from here remembering the definition of $F(z)$ (612) and by setting $z = 0$.

Let us apply the Poisson resummation formula (610) to the sum over n' in (602):

$$Z_{\text{inst}} = \sum_n e^{-\frac{\pi R^2}{2\tau_2}\tau\bar{\tau}n^2} \sum_{n'} e^{-\frac{\pi R^2}{2\tau_2}(n'^2 - n'n(\tau+\bar{\tau}))} \tag{618}$$

$$\int_{-\infty}^{\infty} e^{-\frac{\pi R^2}{2\tau_2}(x^2 - xn(\tau+\bar{\tau}))} e^{2\pi i x p} dx = \frac{\sqrt{2\tau_2}}{R} e^{\left[\frac{\pi R^2}{8\tau_2} n^2(\tau+\bar{\tau})^2 + 2inp\pi\tau_1 - \frac{2p^2\pi\tau_2}{R^2}\right]} \tag{619}$$

Hence we have

$$\begin{aligned} Z_{\text{inst}} &= \frac{\sqrt{2\tau_2}}{R} \sum_n e^{-\frac{\pi R^2}{2\tau_2}\tau\bar{\tau}n^2} \sum_m e^{\left[\frac{\pi R^2}{8\tau_2} n^2(\tau+\bar{\tau})^2 + 2inm\pi\tau_1 - \frac{2m^2\pi\tau_2}{R^2}\right]} \\ &= \frac{\sqrt{2\tau_2}}{R} \sum_{n,m} e^{\left[-\frac{1}{2}\pi R^2 n^2 \tau_2 + 2inm\pi\tau_1 - \frac{2m^2\pi\tau_2}{R^2}\right]} \\ &= \frac{\sqrt{2\tau_2}}{R} \sum_{m,n} q^{\frac{1}{2}\left(\frac{m}{R} + \frac{nR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{m}{R} - \frac{nR}{2}\right)^2} \end{aligned} \tag{620}$$

Collecting all we have

$$Z = \frac{1}{\eta\bar{\eta}} \sum_{m,n} q^{\frac{1}{2}\left(\frac{m}{R}+\frac{nR}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{m}{R}-\frac{nR}{2}\right)^2} \tag{621}$$

The η factors have clear Hamiltonian interpretation. The bosonic Fock space generated by α_{-k} consists of all states of the form $|m,n\rangle$, $\alpha_{-k}|m,n\rangle$, $\alpha_{-k}^2|m,n\rangle$. Hence calculating trace in the $|m,n\rangle$ sector we obtain:

$$\mathrm{Tr} q^{L_0} = \prod_{k=1}^{\infty}(1+q^k+q^{2k}+\cdots) = \prod_{k=1}^{\infty} \frac{1}{1-q^k} \tag{622}$$

The instanton part is easily obtained from formulas (380) and (381).

$U(1)_k$ theory

The $U(1)_k$ chiral algebra ($k \in Z$) contains, besides the Gaussian $U(1)$ current $J = i\sqrt{2k}\partial X$, two additional generators

$$\Gamma^{\pm} = e^{\pm i\sqrt{2k}X} \tag{623}$$

of integer dimension k and charge $\pm 2k$. The primary fields of the extended theory are those vertex operators $e^{i\gamma X}$ whose OPEs with the generators (623) are local. This fixes γ to be

$$\gamma = \frac{n}{\sqrt{2k}}, \quad n \in Z \tag{624}$$

Their conformal dimension is $\Delta_n = \frac{n^2}{4k}$. For primary fields, the range of n must be restricted to the fundamental domain $n = -k+1, -k+2, \ldots, k$ since a shift of n by $2k$ in $e^{inX/\sqrt{2k}}$ amounts to an insertion of the ladder operator Γ^+, which thereby produces a descendant field.

From the point of view of the extended algebra the characters are easily derived. A factor $q^{\Delta_n - 1/24}/\eta(q)$ takes care of the action of the free boson generators. To account for the effect of the distinct multiple applications of the generators (623), which yield shifts of the momentum n by integer multiples of $2k$, we must replace n by $n + l2k$ and sum over l. The net result is

$$\psi_n(q) = \frac{1}{\eta(q)} \sum_{l \in Z} q^{k(l+n/2k)^2}. \tag{625}$$

The action of the modular transformation S on the characters (625) is

$$\psi_n(q') = \frac{1}{\sqrt{2k}} \sum_{n'} e^{\frac{-i\pi n n'}{k}} \psi_{n'}(q) \quad q = e^{2\pi i\tau} \quad \tau' = -\frac{1}{\tau}. \tag{626}$$

Lecture 15
Crossing symmetry and Fusion matrix

Consider the for 4-point correlation function $\langle \Phi_i(\infty)\Phi_k(1,1)\Phi_j(z,\bar{z})\Phi_l(0,0)\rangle$. Using OPE:

$$\Phi_{(j\bar{j})}(z,\bar{z})\Phi_{(l\bar{l})}(0,0) = \sum_{p,\bar{p}} C^{(p\bar{p})}_{(j\bar{j})(l\bar{l})} z^{h_p-h_j-h_l}\bar{z}^{\bar{h}_p-\bar{h}_j-\bar{h}_l}\Psi_{p,\bar{p}}(z,\bar{z}|0,0) \tag{627}$$

where

$$\Psi_{p,\bar{p}}(z,\bar{z}|0,0) = \sum_{k,\bar{k}} \beta^{p,k}_{jl}\bar{\beta}^{\bar{p},\bar{k}}_{\bar{j}\bar{l}} z^K \bar{z}^{\bar{K}} \phi^{k,\bar{k}}_{p,\bar{p}}(0,0) \tag{628}$$

where $K = \sum k_i$.

one can write

$$\sum_{p\bar{p}} C^{p\bar{p}}_{j\bar{j}l\bar{l}} C^{i\bar{i}}_{k\bar{k}p\bar{p}} \mathcal{F}_p \begin{bmatrix} k & j \\ i & l \end{bmatrix}(z) \mathcal{F}_{\bar{p}} \begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix}(\bar{z}) \tag{629}$$

where

$$\mathcal{F}_p \begin{bmatrix} k & j \\ i & l \end{bmatrix}(z) = z^{h_p-h_j-h_l}\sum_k \beta^{p,k}_{jl} z^K \frac{\langle h_i|\Phi_k(1)L_{-k_1}\cdots L_{-k_N}|h_p\rangle}{\langle h_i|\Phi_k(1)|h_p\rangle} \tag{630}$$

is so called conformal block. This conformal block is normalized

$$\lim_{z\to 0}\mathcal{F}_p \begin{bmatrix} k & j \\ i & l \end{bmatrix}(z) = z^{h_p-h_j-h_l} + \cdots \tag{631}$$

By conformal transformation $z \to 1-z$ we can write the correlation function in the form

$$\sum_{q\bar{q}} C^{q\bar{q}}_{k\bar{k}j\bar{j}} C^{i\bar{i}}_{q\bar{q}l\bar{l}} \mathcal{F}_q \begin{bmatrix} l & j \\ i & k \end{bmatrix}(1-z) \mathcal{F}_{\bar{q}} \begin{bmatrix} \bar{l} & \bar{j} \\ \bar{i} & \bar{k} \end{bmatrix}(1-\bar{z}), \tag{632}$$

These two conformal blocks are related by the fusing matrix

$$\mathcal{F}_p \begin{bmatrix} k & j \\ i & l \end{bmatrix}(z) = \sum_q F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}_q \begin{bmatrix} l & j \\ i & k \end{bmatrix}(1-z), \tag{633}$$

and hence one has:

$$\sum_{p\bar{p}} C^{p\bar{p}}_{j\bar{j}l\bar{l}} C^{i\bar{i}}_{k\bar{k}p\bar{p}} F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix} F_{\bar{p},\bar{q}} \begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix} = C^{q\bar{q}}_{k\bar{k}j\bar{j}} C^{i\bar{i}}_{q\bar{q}l\bar{l}}\,. \tag{634}$$

Using the relation

$$\sum_{\bar{q}} F_{\bar{p},\bar{q}^*} \left[\begin{array}{cc} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{array} \right] F_{\bar{q},s} \left[\begin{array}{cc} \bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^* \end{array} \right] = \delta_{\bar{p}s}, \tag{635}$$

Eq. (634) can be written in the form:

$$\sum_p C^{p\bar{p}}_{j\bar{j}l\bar{l}} C^{i\bar{i}}_{k\bar{k}p\bar{p}} F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = \tag{636}$$
$$\sum_{\bar{q}} C^{q\bar{q}}_{k\bar{k}j\bar{j}} C^{i\bar{i}}_{q\bar{q}l\bar{l}} F_{\bar{q}^*,\bar{p}} \left[\begin{array}{cc} \bar{j} & \bar{l} \\ \bar{k}^* & \bar{i}^* \end{array} \right].$$

Putting in (634) $i = \bar{i} = 0$ we obtain the following useful relation:

$$C^{k^*,\bar{k}^*}_{j\bar{j}l\bar{l}} C^0_{k\bar{k},k^*\bar{k}^*} = C^{l^*\bar{l}^*}_{k\bar{k}j\bar{j}} C^0_{l^*\bar{l}^*,l\bar{l}}\,. \tag{637}$$

For diagonal model

$$C^{p\bar{p}}_{k\bar{k}i\bar{i}} = C^p_{ki} \delta_{\bar{p}p^*} \delta_{\bar{k}k^*} \delta_{\bar{i}i^*} \tag{638}$$

Eq. (636) takes the form:

$$C^i_{kp} C^p_{jl} F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = C^q_{kj} C^i_{ql} F_{q,p} \left[\begin{array}{cc} k^* & i \\ j & l^* \end{array} \right]. \tag{639}$$

To derive (639) we also used the symmetry properties

$$F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = F_{p^*,q} \left[\begin{array}{cc} j & k \\ l^* & i^* \end{array} \right] = F_{p,q^*} \left[\begin{array}{cc} i^* & l \\ k^* & j \end{array} \right] = F_{p^*,q^*} \left[\begin{array}{cc} l & i^* \\ j^* & k \end{array} \right]. \tag{640}$$

Using (637), (639) takes the form

$$C^{p^*}_{ki^*} C^p_{jl} C^0_{pp^*} F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = C^q_{kj} C^{q^*}_{i^*l} C^0_{qq^*} F_{q,p} \left[\begin{array}{cc} k^* & i \\ j & l^* \end{array} \right]. \tag{641}$$

To derive (641) we used (637) and the commutativity of the structure constants by two lower indices in diagonal models :

$$C^j_{ik,c\bar{c}} = C^j_{ki,c\bar{c}}\,. \tag{642}$$

Setting $q = 0$, $k = j^*$, $i = l$ in (641), and using

$$C^0_{ii^*} = \frac{C_{ii^*}}{C_{00}}\,, \tag{643}$$

where C_{ii^*} are two-point functions, we obtain:

$$\left(C_{ij}^{p}\right)^2 = \frac{C_{jj^*}C_{ii^*}F_{0,p}\begin{bmatrix} j & i \\ j & i^* \end{bmatrix}}{C_{00}C_{pp^*}F_{p,0}\begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}\,. \tag{644}$$

Using the relation

$$F_{0,i}\begin{bmatrix} j & k \\ j & k^* \end{bmatrix} F_{i,0}\begin{bmatrix} k^* & k \\ j & j \end{bmatrix} = \frac{F_j F_k}{F_i}\,, \tag{645}$$

where

$$F_i \equiv F_{0,0}\begin{bmatrix} i & i^* \\ i & i \end{bmatrix}\,. \tag{646}$$

we can write (644) in two forms

$$C_{ij}^{p} = \frac{\eta_i\eta_j}{\eta_0\eta_p}F_{0,p}\begin{bmatrix} j & i \\ j & i^* \end{bmatrix}\,, \tag{647}$$

and

$$C_{ij}^{p} = \frac{\xi_i\xi_j}{\xi_0\xi_p}\frac{1}{F_{p,0}\begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}\,, \tag{648}$$

where

$$\eta_i = \sqrt{C_{ii^*}/F_i}\,, \tag{649}$$

and

$$\xi_i = \eta_i F_i = \sqrt{C_{ii^*}F_i}\,. \tag{650}$$

Lecture 16
Conformal blocks of the Ising model

Consider correlation functions requiring a single screening operator. For instance:

$$\langle V_{\overline{n,m}} V_{1,2} V_{1,2} V_{n,m} \rangle \tag{651}$$

has charge $2\alpha_0 - \alpha_-$ and therefore requires one Q_-, and

$$\langle V_{\overline{n,m}} V_{2,1} V_{2,1} V_{n,m} \rangle \tag{652}$$

has charge $2\alpha_0 - \alpha_+$ and therefore requires one Q_+.

Hence these conformal blocks have general form:

$$\oint dw \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) V_\pm(w) \rangle \tag{653}$$

We can write this block in a more canonical form using $SL(2,C)$ invariance

$$\oint dw \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) V_\pm(w) \rangle = \tag{654}$$

$$\oint dw \prod_{i=1}^{4} (cz_i + d)^{-2h_i} \langle \prod_{i=1}^{4} V_{\alpha_i} \left(\frac{az_i + b}{cz_i + d} \right) V_\pm(w) \rangle$$

Since w is integrated and $V_\pm(w)$ is a 1-form, we can forget about the w transformation. We now choose a, b, c, d, so that $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to \eta$, $z_4 \to 0$, where

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \tag{655}$$

This is achieved by

$$w = \frac{(z - z_4)(z_1 - z_2)}{(z_1 - z)(z_2 - z_4)} \tag{656}$$

leading to

$$\oint dw \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) V_\pm(w) \rangle \tag{657}$$

$$= \left(\frac{z_{12} z_{14}}{z_{24}} \right)^{h_2 + h_3 + h_4 - h_1} \frac{(1-\eta)^{2\alpha_2\alpha_3} \eta^{2\alpha_3\alpha_4}}{z_{12}^{2h_2} z_{13}^{2h_3} z_{14}^{2h_4}} \oint dw (1-w)^{2\alpha_2\alpha_\pm} (\eta - w)^{2\alpha_3\alpha_\pm} w^{2\alpha_4\alpha_\pm}$$

There are two independent contours $C_1 = [0, \eta]$ and $C_2 = [1, \infty[$, leading to the two functions:

$$\oint_0^\eta dw (1-w)^\alpha (\eta - w)^\beta w^\gamma \tag{658}$$

$$= \frac{\Gamma(1+\gamma)\Gamma(1+\beta)}{\Gamma(2+\gamma+\beta)} \eta^{1+\beta+\gamma} F(-\alpha, 1+\gamma, 2+\gamma+\beta; \eta)$$

$$\oint_1^\infty dw(1-w)^\alpha(\eta-w)^\beta w^\gamma \tag{659}$$
$$= \frac{\Gamma(1+\alpha)\Gamma(-\alpha-\beta-\gamma-1)}{\Gamma(-\gamma-\beta)} F(-\beta,-\alpha-\beta-\gamma-1,-\gamma-\beta;\eta)$$

We need the following properties of hypergeometric function:

$$F(a,b,c,;z) = (1-z)^{c-a-b}F(c-a,c-b,c,;z) \tag{660}$$

$$F(-n,n,c;z) = \frac{1}{(c)_n}z^{1-c}(1-z)^{n-b+c}\frac{d^n}{dz^n}[z^{n+c-1}(1-z)^{b-c}] \tag{661}$$

where $(c)_n = c(c+1)\cdots(c+n-1)$.

$$\cos az = \cos zF\left(\frac{1}{2}+\frac{a}{2},\frac{1}{2}-\frac{a}{2},\frac{1}{2};\sin^2 z\right) \tag{662}$$

$$\sin az = a\cos z\sin zF\left(1+\frac{a}{2},1-\frac{a}{2},\frac{3}{2};\sin^2 z\right) \tag{663}$$

Let us consider the four-spin correlator $\langle\sigma\sigma\sigma\sigma\rangle$.

In this case $\alpha_1=\alpha_2=\alpha_3=\alpha_4=-\frac{\alpha_-}{2}$. The conformal blocks are given by the integrals (658) and (659) with $\alpha=\beta=\gamma=2\alpha_i\alpha_-=-\alpha_-^2=-\frac{3}{4}$. The corresponding functions are:

$$\frac{1}{\sqrt{\eta}}F\left(\frac{3}{4},\frac{1}{4},\frac{1}{2},\eta\right) \tag{664}$$

and

$$F\left(\frac{3}{4},\frac{5}{4},\frac{3}{2},\eta\right) \tag{665}$$

Using (662) and (663) we obtain for (664)

$$\sqrt{\frac{1+\sqrt{1-\eta}}{2\eta(1-\eta)}} \tag{666}$$

and for (665)

$$\frac{1}{2}\sqrt{\frac{1-\sqrt{1-\eta}}{2\eta(1-\eta)}} \tag{667}$$

Multiplying also by $(1-\eta)^{3/8}\eta^{3/8}$ in front of integral we get for (664) :

$$\frac{1}{\eta^{1/8}(1-\eta)^{1/8}}\sqrt{\frac{1+\sqrt{1-\eta}}{2}} \tag{668}$$

and for (665)

$$\frac{1}{\eta^{1/8}(1-\eta)^{1/8}}\frac{1}{2}\sqrt{\frac{1-\sqrt{1-\eta}}{2}} \tag{669}$$

Now we should fix normalizations. In the limit $\eta \to 0$ in (668) yields $\eta^{1/8}$. Therefore the normalization is correct. In the same limit (669) should yields $\eta^{3/8}$. Therefore the correctly normalized block is

$$\frac{\sqrt{2}}{\eta^{1/8}(1-\eta)^{1/8}}\sqrt{1-\sqrt{1-\eta}} \tag{670}$$

Thus we obtained for conformal blocks:

$$\mathcal{F}_I = \frac{1}{\eta^{1/8}(1-\eta)^{1/8}}\sqrt{\frac{1+\sqrt{1-\eta}}{2}} \tag{671}$$

and

$$\mathcal{F}_\epsilon = \frac{\sqrt{2}}{\eta^{1/8}(1-\eta)^{1/8}}\sqrt{1-\sqrt{1-\eta}} \tag{672}$$

The full correlation function is

$$\begin{aligned} &\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle \qquad (673) \\ &= \frac{1}{|z_{13}z_{24}\eta(1-\eta)|^{1/4}}\left(\frac{1}{2}(C^I_{\sigma\sigma})^2|1+\sqrt{1-\eta}| + 2(C^\epsilon_{\sigma\sigma})^2|1-\sqrt{1-\eta}|\right) \end{aligned}$$

The structure constant $C^I_{\sigma\sigma}$ is fixed by normalization to be 1. The other structure constant $C^\epsilon_{\sigma\sigma}$ can be found from the requirement of the (673) to be invariant under fusion. To check the invariance it is convenient to introduce the variable $\eta = \sin^2 z$ and write the expression in the parenthesis in the form:

$$\cos\frac{z}{2}\cos\frac{\bar{z}}{2} + 4(C^\epsilon_{\sigma\sigma})^2\sin\frac{z}{2}\sin\frac{\bar{z}}{2} \tag{674}$$

The transformation $\eta \to 1-\eta$ takes the form $z \to \frac{\pi}{2} - z$. For $C^\epsilon_{\sigma\sigma} = \frac{1}{2}$ (674) reads

$$\cos\left(\frac{z-\bar{z}}{2}\right) \tag{675}$$

which obviously invariant under $z \to \frac{\pi}{2} - z$.

Let us find fusion matrix. The relevant transforming part of conformal blocks using the z variable are

$$f_I = \cos\frac{z}{2} \tag{676}$$

and

$$f_\epsilon = 2\sin\frac{z}{2} \tag{677}$$

These functions under the transformation $z \to \frac{\pi}{2} - z$ transform in the following way

$$f_I\left(\frac{\pi}{2} - z\right) = \frac{1}{\sqrt{2}}\left(f_I + \frac{1}{2}f_\epsilon\right) \tag{678}$$

$$f_\epsilon\left(\frac{\pi}{2} - z\right) = \frac{1}{\sqrt{2}}\left(2f_I - f_\epsilon\right) \tag{679}$$

leading to the following elements of fusion matrix

$$F_{II}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{1}{\sqrt{2}} \quad F_{I\epsilon}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{1}{2}\frac{1}{\sqrt{2}} \tag{680}$$

$$F_{\epsilon I}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = \frac{2}{\sqrt{2}} \quad F_{\epsilon\epsilon}\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} = -\frac{1}{\sqrt{2}} \tag{681}$$

Collecting all we obtain for correlation function

$$\begin{aligned} &\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3)\sigma(z_4, \bar{z}_4)\rangle \\ &= \frac{1}{2}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{1/4}\left(|1 + \sqrt{1-\eta}| + |1 - \sqrt{1-\eta}|\right) \end{aligned} \tag{682}$$

This can also be written in the form:

$$\begin{aligned} &\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\sigma(z_3, \bar{z}_3)\sigma(z_4, \bar{z}_4)\rangle \\ &= \frac{1}{\sqrt{2}}\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{1/4}\left(1 + |\eta| + |1-\eta|\right)^{1/2} \end{aligned} \tag{683}$$

Lecture 17
Topological preliminaries

Vector fields-Tangent space

$$X(fg) = Xf \cdot g + f \cdot Xg \tag{684}$$

$$Xf = X^i \frac{\partial f}{\partial x^i} \tag{685}$$

$$[X,Y]f = X(Yf) - Y(Xf) \tag{686}$$

Cotangent space

$$\alpha(X) = \alpha_i X^i \tag{687}$$

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j \tag{688}$$

$$\alpha = \alpha_i dx^i \tag{689}$$

$$df(X) = Xf = X^i \frac{\partial f}{\partial x^i} \tag{690}$$

This implies

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{691}$$

Differential form:

$$\omega(X_{\sigma(i_1)}, \cdots X_{\sigma(i_n)}) = \epsilon_\sigma \omega(X_1, \cdots X_n) \tag{692}$$

$$dx^{i_1} \wedge \cdots \wedge dx^{i_n} = \sum_\sigma \epsilon_\sigma \left(dx^{\sigma(i_1)} \wedge \cdots \wedge dx^{\sigma(i_n)}\right) \tag{693}$$

The expression (693) is antisymmetric towards permutation of indices

$$dx^{\sigma(i_1)} \wedge \cdots \wedge dx^{\sigma(i_n)} = \epsilon_\sigma dx^{i_1} \wedge \cdots \wedge dx^{i_n} \tag{694}$$

The expression (693) form the basis in the space of the antisymmetric tensors:

$$\omega = \frac{1}{n!}\omega_{i_1 \dots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \tag{695}$$

Dimension is $C^n_N = \frac{N!}{n!(N-n)!}$.

The wedge product

$$\alpha = \frac{1}{n!}\alpha_{i_1 \dots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \tag{696}$$

$$\beta = \frac{1}{m!}\beta_{j_1 \dots j_m} dx^{j_1} \wedge \cdots \wedge dx^{j_m} \tag{697}$$

$$\alpha \wedge \beta = \frac{1}{n!m!} \alpha_{i_1 \dots i_n} \beta_{j_1 \dots j_m} dx^{i_1} \wedge \dots \wedge dx^{i_n} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m} \tag{698}$$

$$(\alpha \wedge \beta)(X_1, \dots X_{n+m}) = \sum_{\sigma} \epsilon_{\sigma} \alpha(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \beta(X_{\sigma(n+1)}, \dots, X_{\sigma(n+m)}) \tag{699}$$

where sum runs over all permutations σ with the property

$$\sigma(1) < \dots < \sigma(n) \qquad \text{and} \qquad \sigma(n+1) < \dots < \sigma(n+m) \tag{700}$$

Exterior derivative

$$d\omega = \frac{1}{n!} d\omega_{i_1 \dots i_n} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n} = \frac{1}{n!} \frac{\partial \omega_{i_1 \dots i_n}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n} \tag{701}$$

In components

$$d\omega_{j_1 \dots j_{n+1}} = \sum_{a=1}^{n+1} (-)^{a+1} \frac{\partial \omega_{j_1 \dots \hat{j}_a \dots j_{n+1}}}{\partial x^{j_a}} \tag{702}$$

Hat means here omitted.

We also have

$$\begin{aligned} & d\omega(X_1, \dots X_{m+1}) = \\ & \sum_{a=1}^{m+1} (-)^{a+1} X_a \omega(X_1, \dots, \hat{X}_a, \dots, X_{m+1}) + \\ & + \sum_{a=1}^{m} \sum_{b=a+1}^{m+1} (-)^{a+b} \omega([X_a, X_b], X_1, \dots, \hat{X}_a, \dots, \hat{X}_a, \dots, X_{m+1}) \end{aligned} \tag{703}$$

The exterior derivative has the following properties:

$$d^2 = 0 \tag{704}$$

$$d(\theta \wedge \omega) = d\theta \wedge \omega + (-)^m \theta \wedge d\omega \tag{705}$$

where m is the degree of θ.

Pullback

Assume we have map of two manifolds $F : \mathcal{X} \to \mathcal{Y}$.

Then we have map of $C^{\infty}(\mathcal{Y}) \to C^{\infty}(\mathcal{X})$ given by

$$f_X(x) = f_Y(F(x)) = f_Y \circ F(x) \tag{706}$$

Map (706) defines the map of vector fields on $\mathcal{X}$ to vector fields on $\mathcal{Y}$m called differential of the F

$$Y = dF(X)(f_Y) = X(f_Y \circ F) \tag{707}$$

Now we can define pullback of the differential form, mapping $F^* : \Omega(\mathcal{Y}) \to \Omega(\mathcal{X})$

$$F^*\omega(X_1, \ldots X_m) = \omega(dF(X_1), \ldots dF(X_m)) \tag{708}$$

where ω is a form of degree m on $\mathcal{Y}$.

We can write the map F in the local coordinates as

$$y^j = F^j(x_1, \ldots x_N), \qquad j = 1, \ldots M \tag{709}$$

Then the differential map in components take the form:

$$Y^j = \frac{\partial F^j}{\partial x^i} X^i \tag{710}$$

The pullback form has components:

$$(F^*\omega)_{i_1 \ldots i_m} = \frac{\partial F^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial F^{j_m}}{\partial x^{i_m}} \omega_{j_1 \ldots j_m} \circ F \tag{711}$$

The exterior derivative commutes with the pullback map:

$$dF^*\omega = F^* d\omega \tag{712}$$

The form satisfying

$$d\omega = 0 \tag{713}$$

is called closed. The form satisfying

$$\omega = d\alpha \tag{714}$$

is exact. Every exact form is closed, but vice verse in general is not true. Consider the exterior derivative acting on forms Ω^m of degree m. The factor of closed forms by exact is called cohomology of degree m:

$$H^m = \mathrm{ker} d / \mathrm{Im} d \tag{715}$$

Integration

Consider the form:

$$\omega = \frac{1}{n!} \omega_{i_1 \ldots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \tag{716}$$

In the different coordinate system $x^{i'}$ it has the form:

$$\omega = \frac{1}{n!}\omega_{i'_1\dots i'_n} dx^{i'_1} \wedge \dots \wedge dx^{i'_n} \tag{717}$$

with the components:

$$\omega_{i_1\dots i_n} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_n}}{\partial x^{i_n}} \omega_{i'_1\dots i'_n} \tag{718}$$

Consider the differential form of the maximum degree N, equal to the dimension of the space. It has one component

$$\omega = \rho dx^1 \wedge \dots \wedge x^N \tag{719}$$

which transforms as a density, namely gets multiplied by the Jacobian:

$$\rho' = \rho J \tag{720}$$

Therefore the integral

$$\int_{\mathcal{X}} \rho dx^1 \wedge \dots \wedge x^N \tag{721}$$

is independent on the change of the coordinates.

Stoks theorem

If the manifold has a boundary we have the Stoks theorem:

$$\int_{\mathcal{X}} d\omega = \int_{\partial\mathcal{X}} d\omega \tag{722}$$

Homotopy groups

Two smooth maps f and g between $\mathcal{X}$ and $\mathcal{Y}$ are homotopic if there exist a smooth map

$$F : \mathcal{X} \times I \to \mathcal{Y} \qquad I = [0, 1] \tag{723}$$

such that

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(x, 1) = g(x) \tag{724}$$

By other words f and g homotopic if can be smoothly deformed to each other via the family of maps $f_t = F(x, t)$. Under the relation of the homotopy all maps between $\mathcal{X}$ and $\mathcal{Y}$ divided to classes, homotopy classes. The homotopy classes of the maps of sphere S^n to a manifold called group of homotopy $\pi_n(M)$.

Lecture 18
WZW model-Action

The world-sheet action of the bulk WZW model is

$$\begin{aligned} S^{\mathrm{WZW}}(g) &= \frac{k}{4\pi}\int_\Sigma \mathrm{Tr}(\partial_z g^{-1}\partial_{\bar z} g)dzd\bar z + \frac{k}{4\pi}\int_B \frac{1}{3}\mathrm{tr}(g^{-1}dg)^3 \\ &\equiv \frac{k}{4\pi}\left[\int_\Sigma dzd\bar z L^{\mathrm{kin}} + \int_B \omega^{\mathrm{WZ}}\right], \end{aligned} \tag{725}$$

B is a 3-manifold such that $\partial B = \Sigma$. This action depends on the extension of the field on three-manifold B. However this extension is not unique, hence there is a potential ambiguity in the definition of the second term. Indeed, in a compactified three-dimensional space, a compact two-dimensional space delimits two distinct three-manifolds. The difference between two choices quantifies the ambiguity. Taking the orientation into account, the difference is given by the second term but with the integration range extended over the whole compact three-dimensional space. Since the latter is topological equivalent to the three-sphere one can write

$$\Delta S^{\mathrm{WZW}} = \int_{S^3} \omega^{\mathrm{WZ}} \tag{726}$$

Now we show that this integral is integer. First of all using the relation

$$\delta\omega^{WZ} = d[\mathrm{Tr}(\delta g g^{-1}(dgg^{-1})^2)] \tag{727}$$

we obtain:

$$\int_{S^3} \delta\omega^{\mathrm{WZ}} = 0 \tag{728}$$

Hence the integral (726) is invariant under the continuous deformation of g. It implies that the integral depends only on the homotopy class of g. On the other hand the Polyakov-Wiegmann identity

$$\omega^{\mathrm{WZ}}(gh) = \omega^{\mathrm{WZ}}(g) + \omega^{\mathrm{WZ}}(h) - \mathrm{d}\Big(\mathrm{Tr}(g^{-1}\mathrm{d}g\mathrm{d}hh^{-1})\Big), \tag{729}$$

implies

$$\int_{S^3} \omega^{\mathrm{WZ}}(gh) = \int_{S^3} \omega^{\mathrm{WZ}}(g) + \int_{S^3} \omega^{\mathrm{WZ}}(h) \tag{730}$$

Let us also recall that $\pi_3(G) = \mathbb{Z}$. Therefore homotopy classes of the map $g : S^3 \to G$ are labelled by integer numbers, and maps belonging to different classes have different value of the integral (726). Choosing the map g_1 with the unit value of the integral we can take as representative of other classes the maps g_1^n. Each such representative has value n of (726). Therefore with k integer the functional integral is well defined.

We can also show that for $G = SU(2)$ the integral (726) coincide with degree of map.

$$\begin{aligned}
&\delta(\mathrm{Tr}(\partial_z g^{-1}\partial_{\bar z} g)) = \qquad (731)\\
&\mathrm{Tr}\Big(\delta g g^{-1}[2\partial_z\partial_{\bar z} g g^{-1} - \partial_{\bar z} g g^{-1}\partial_z g g^{-1} - \partial_z g g^{-1}\partial_{\bar z} g g^{-1}]\Big)\\
&-\mathrm{Tr}\Big(\partial_z(\delta g g^{-1}\partial_{\bar z} g g^{-1}) + \partial_{\bar z}(\delta g g^{-1}\partial_z g g^{-1})\Big)\\
&= \mathrm{Tr}\Big(\delta g g^{-1}[\partial_{\bar z}(\partial_z g g^{-1}) + \partial_z(\partial_{\bar z} g g^{-1})] - \partial_z(\delta g g^{-1}\partial_{\bar z} g g^{-1}) - \partial_{\bar z}(\delta g g^{-1}\partial_z g g^{-1})\Big)
\end{aligned}$$

$$\delta\omega^{WZ} = d[\mathrm{Tr}(\delta g g^{-1}(dg g^{-1})^2)] \qquad (732)$$

$$\begin{aligned}
\int_B \delta\omega^{WZ} &= \int_\Sigma \mathrm{Tr}(\delta g g^{-1}[\partial_{\bar z} g g^{-1}\partial_z g g^{-1} - \partial_z g g^{-1}\partial_{\bar z} g g^{-1}] \qquad (733)\\
&= \int_\Sigma \mathrm{Tr}(\delta g g^{-1}[\partial_{\bar z}(\partial_z g g^{-1}) - \partial_z(\partial_{\bar z} g g^{-1})]
\end{aligned}$$

Taking the sum of (731) and (733) and omitting the full derivative terms we obtain:

$$\delta S^{\mathrm{WZW}}(g) = \frac{k}{2\pi}\int_\Sigma dz d\bar z \mathrm{Tr}[\delta g g^{-1}\partial_{\bar z}(\partial_z g g^{-1})] \qquad (734)$$

Alternatively we can write:

$$\begin{aligned}
&\delta(\mathrm{Tr}(\partial_z g^{-1}\partial_{\bar z} g)) = \qquad (735)\\
&= \mathrm{Tr}\Big[g^{-1}\delta g[\partial_{\bar z}(g^{-1}\partial_z g) + \partial_z(g^{-1}\partial_{\bar z} g)] - \partial_z(\delta g g^{-1}\partial_{\bar z} g g^{-1}) - \partial_{\bar z}(\delta g g^{-1}\partial_z g g^{-1})\Big]
\end{aligned}$$

$$\delta\omega^{WZ} = d[\mathrm{Tr}(g^{-1}\delta g(g^{-1}dg)^2)] \qquad (736)$$

$$\begin{aligned}
\int_B \delta\omega^{WZ} &= \int_\Sigma \mathrm{Tr}(g^{-1}\delta g[g^{-1}\partial_{\bar z} g g^{-1}\partial_z g - g^{-1}\partial_z g g^{-1}\partial_{\bar z} g] \qquad (737)\\
&= \int_\Sigma \mathrm{Tr}(g^{-1}\delta g[\partial_z(g^{-1}\partial_{\bar z} g) - \partial_{\bar z}(g^{-1}\partial_z g)]
\end{aligned}$$

Again taking the sum of (735) and (737) and omitting the full derivative terms we obtain:

$$\delta S^{\mathrm{WZW}}(g) = \frac{k}{2\pi}\int_\Sigma dz d\bar z \mathrm{Tr}[g^{-1}\delta g\partial_z(g^{-1}\partial_{\bar z} g)] \qquad (738)$$

Let us draw conclusions from equations (734) and (738). Taking $\delta g g^{-1}$or $g^{-1}\delta g$ arbitrary we get that EOM of the WZW model is

$$\partial_{\bar{z}}(\partial_z g g^{-1}) = 0 \tag{739}$$

or equivalently

$$\partial_z(g^{-1}\partial_{\bar{z}} g) = 0 \tag{740}$$

On the other hand taking $\delta g g^{-1} \equiv \omega(z)$ holomorphic we see from (734) using the integration by parts that $\delta_\omega S = 0$ identically. Therefore the WZW action (725) has the symmetry

$$\delta g = \omega(z) g \tag{741}$$

and the corresponding conserved current is

$$J(z) = -k\partial_z g g^{-1} \tag{742}$$

The EOM in the form (739) coincides with the condition of of the conservation of the current (742). Therefore the current (742) is holomorphic. Similarly taking $g^{-1}\delta g \equiv \bar{\omega}(\bar{z})$ anti-holomorphic we receive from (738) using the integration by parts that $\delta_{\bar{\omega}} S = 0$ identically. Hence the action (725) has additionally the symmetry

$$\delta_{\bar{\omega}} g = -g\bar{\omega}(\bar{z}) \tag{743}$$

and the corresponding conserved current is

$$\bar{J}(z) = k g^{-1}\partial_{\bar{z}} g \tag{744}$$

Again the EOM in the form (740) coincides with the condition of of the conservation of the current (744). Therefore the current (744) is anti-holomorphic. Classically the components of the tensor-energy momentum are

$$T = \frac{1}{2k}\mathrm{Tr} J^2 \tag{745}$$

$$\bar{T} = \frac{1}{2k}\mathrm{Tr} \bar{J}^2 \tag{746}$$

The symmetries of the WZW model can be also derived using the Polyakov-Wiegmann identities:

$$L^{\mathrm{kin}}(gh) = L^{\mathrm{kin}}(g) + L^{\mathrm{kin}}(h) - \Big(\mathrm{Tr}(g^{-1}\partial_z g \partial_{\bar{z}} h h^{-1}) + \mathrm{Tr}(g^{-1}\partial_{\bar{z}} g \partial_z h h^{-1})\Big) \tag{747}$$

$$\omega^{\mathrm{WZ}}(gh) = \omega^{\mathrm{WZ}}(g) + \omega^{\mathrm{WZ}}(h) - \mathrm{d}\Big(\mathrm{Tr}(g^{-1}\mathrm{d}g\mathrm{d}hh^{-1})\Big)\,, \tag{748}$$

Let us elaborate the WZW action for $SU(2)$ group. A three-sphere S^3 is a group manifold of the $SU(2)$ group. A generic element in this group can be written as

$$g = X_0\sigma_0 + i(X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3) = \begin{pmatrix} X_0 + iX_3 & X_2 + iX_1 \\ -(X_2 - iX_1) & X_0 - iX_3 \end{pmatrix} \tag{749}$$

subject to condition that the determinant is equal to one

$$X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1\,. \tag{750}$$

The metric on S^3 can be written in the following three ways, which will be used in the main text. Firstly, using the Euler parametrisation of the group element we have

$$g = e^{i\chi\frac{\sigma_3}{2}} e^{i\tilde{\theta}\frac{\sigma_1}{2}} e^{i\varphi\frac{\sigma_3}{2}} \tag{751}$$

$$\mathrm{d}s^2 = \frac{1}{4}\Big((\mathrm{d}\chi + \cos\tilde{\theta}\mathrm{d}\varphi)^2 + \mathrm{d}\tilde{\theta}^2 + \sin^2\tilde{\theta}\mathrm{d}\varphi^2\Big) = \frac{1}{4}\Big(\mathrm{d}\chi^2 + \mathrm{d}\varphi^2 + \mathrm{d}\tilde{\theta}^2 + 2\cos\tilde{\theta}\mathrm{d}\chi\mathrm{d}\varphi\Big) \tag{752}$$

The ranges of coordinates are $0 \leq \tilde{\theta} \leq \pi$, $0 \leq \varphi \leq 2\pi$ and $0 \leq \chi \leq 4\pi$.

Secondly, we can use coordinates that are analogue to the global coordinate for AdS_3

$$X_0 + iX_3 = \cos\theta e^{i\tilde{\phi}}\,, \quad X_2 + iX_1 = \sin\theta e^{i\phi} \tag{753}$$

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \cos^2\theta\mathrm{d}\tilde{\phi}^2 + \sin^2\theta\mathrm{d}\phi^2\,. \tag{754}$$

The relation between the metrics (751) and (753) is given by

$$\chi = \tilde{\phi} + \phi\,, \quad \varphi = \tilde{\phi} - \phi\,, \quad \theta = \frac{\tilde{\theta}}{2}\,. \tag{755}$$

The ranges of coordinates are $-\pi \leq \tilde{\phi}, \phi \leq \pi$ and $0 \leq \theta \leq \frac{\pi}{2}$.

Thirdly, the standard metric on S^3 is given by ($\vec{n}$ is a unit vector on S^2)

$$g = e^{2i\psi\frac{\vec{n}\cdot\vec{\sigma}}{2}}\,, \quad \mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\xi^2 + \sin^2\xi\mathrm{d}\eta^2) \tag{756}$$

$$X_0 + iX_3 = \cos\psi + i\sin\psi\cos\xi\,, \quad X_2 + iX_1 = \sin\psi\sin\xi e^{i\eta}\,. \tag{757}$$

The ranges of the coordinates are $0 \leq \psi\,,\, \xi \leq \pi$ and $0 \leq \eta \leq 2\pi$.

In the parametrisation (751) we have

$$g^{-1}dg = i(L_1\sigma_1 + L_2\sigma_2 + L_3\sigma_3) \tag{758}$$

where

$$\begin{aligned} L_1 &= \frac{1}{2}(-d\tilde{\theta}\sin\varphi + \sin\tilde{\theta}\cos\varphi d\chi) \\ L_2 &= \frac{1}{2}(d\tilde{\theta}\cos\varphi + \sin\tilde{\theta}\sin\varphi d\chi) \\ L_3 &= \frac{1}{2}(d\varphi + \cos\tilde{\theta} d\chi) \end{aligned} \tag{759}$$

$$\mathrm{Tr}(dg^{-1}dg) = 2ds^2 \tag{760}$$

$$\begin{aligned} &\frac{1}{3}\mathrm{Tr}(g^{-1}dg)^3 = -2iL_1 \wedge L_2 \wedge L_3 \mathrm{Tr}\sigma_1\sigma_2\sigma_3 = \\ &4L_1 \wedge L_2 \wedge L_3 = \frac{1}{2}\sin\tilde{\theta} d\chi d\tilde{\theta} d\varphi \end{aligned} \tag{761}$$

Lecture 19
WZW model-Quantization

Remembering (260) we have

$$\delta_{\omega,\bar{\omega}}X = -\frac{1}{2\pi i}\oint dz \sum_a \omega^a J^a X + \frac{1}{2\pi i}\oint d\bar{z} \sum_a \bar{\omega}^a \bar{J}^a X \tag{762}$$

where

$$J = \sum_a J^a T^a, \quad \omega = \sum_a \omega^a T^a, \quad \text{and} \quad \mathrm{Tr}(T^a T^b) = \delta^{ab} \tag{763}$$

The transformation law for the currents follows from (741) and (742)

$$\begin{aligned}\delta_\omega J &= -k(\partial_z(\delta\omega g)g^{-1} - \partial_z g g^{-1}\delta\omega g g^{-1}) \\ &\quad -k(\partial_z \omega g + \omega \partial_z g)g^{-1} + k\partial_z g g^{-1}\omega \\ &\quad [\omega, J] - k\partial_z \omega\end{aligned} \tag{764}$$

It can be rewritten as

$$\delta_\omega J = \sum_{b,c} i f_{abc}\omega^b J^c - k\partial_z \omega^a \tag{765}$$

Comparing (762) and (764) we arrive

$$J^a(z)J^a(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c i f_{abc}\frac{J^c(w)}{(z-w)} \tag{766}$$

This will be called a current algebra. Introducing the modes J_n^a from the Laurent expansion

$$J^a(z) = \sum_{n\in\mathbf{Z}} z^{-n-1}J_n^a \tag{767}$$

we can obtain the commutation relations of the affine algebra Lie at the level k:

$$[J_n^a, J_m^b] = \sum_c i f_{abc}J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0} \tag{768}$$

The transformation property of $\bar{J}$ is

$$\delta_{\bar{\omega}}\bar{J} = [\bar{\omega}, \bar{J}] - k\partial_z\bar{\omega} \tag{769}$$

This yields another copy of the affine algebra for the modes $\bar{J}_m^b$. Since $\bar{\omega}(\bar{z})$ is independent of z

$$\delta_{\bar{\omega}}J = 0 \tag{770}$$

This implies

$$[J^a_n, \bar{J}^b_m] = 0 \tag{771}$$

Normal ordering

The OPE of A and B is written as

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{772}$$

then the normal-ordered version is

$$(AB)(w) = \{AB\}_0(w) \tag{773}$$

The contraction is defined to include all the singular terms of the OPE

$$\overline{A(z)B(w)} = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{774}$$

Hence the above expression for the normal ordered product can be written as:

$$(AB)(w) = \lim_{z\to w}\left[A(z)B(w) - \overline{A(z)B(w)}\right] \tag{775}$$

The method of contour integration provides another useful representation of our newly introduced normal ordering:

$$(AB)(w) = \frac{1}{2\pi i}\oint_w \frac{dz}{z-w} A(z)B(w) \tag{776}$$

$$\overline{A(z)(BC)(w)} = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\{\overline{A(z)B(x)}C(w) + B(x)\overline{A(z)C(w)}\} \tag{777}$$

Consider now the normal ordered version of the tensor energy-momentum:

$$T(z) = \gamma \sum_a (J^a J^a)(z) \tag{778}$$

$$\begin{aligned} J^a(z)(J^aJ^a)(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left[\overline{J^a(z)J^b(x)}J^b(w) + J^b(x)\overline{J^a(z)J^b(w)}\right] = \\ \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\Bigg[\left(\frac{k\delta_{ab}}{(z-x)^2} + \sum_c if_{abc}\frac{J^c(x)}{(z-x)}\right)J^b(w) \\ + J^b(x)\left(\frac{k\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(x)}{(z-w)}\right)\Bigg] \end{aligned} \tag{779}$$

Developing OPE we obtain:

$$J^a(z)(J^aJ^a)(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left[\frac{k\delta_{ab}J^b(w)}{(z-x)^2}\right. \tag{780}$$
$$+\sum_c \frac{if_{abc}}{(z-x)}\left(if_{cbd}\frac{J^d(w)}{(x-w)} + \frac{k\delta_{cb}}{(x-w)^2} + (J^cJ^b)(w)\right)$$
$$\left. + \frac{k\delta_{ab}J^b(w)}{(z-w)^2} + \sum_c if_{abc}\frac{(J^bJ^c)(w)}{(z-w)}\right]$$

Due to the antisymmetry of the structure constant f_{abc} the term $f_{abc}\delta_{cb}$ vanishes. We now sum the result over b and use

$$-\sum_{b,c} f_{abc}f_{cbd} = \sum_{b,c} f_{abc}f_{dbc} = 2h_G\delta_{ad} \tag{781}$$

where h_G is the dual Coxeter number. Moreover we also have

$$\sum_{b,c} f_{abc}[(J^bJ^c) + (J^cJ^b)]0 = \tag{782}$$

We end up with

$$J^a(z)\sum_b (J^bJ^b)(w) = 2(k+h_G)\frac{J^a(w)}{(z-w)^2} \tag{783}$$

Inverting the order of the fields we obtain:

$$T(z)J^a(w) = 2\gamma(k+h_G)\frac{J^a(z)}{(z-w)^2} = 2\gamma(k+h_G)\left[\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}\right] \tag{784}$$

The request of the current to have the weight one leads to the condition:

$$\gamma = \frac{1}{2(k+h_G)} \tag{785}$$

Finally for the tensor energy-momentum we get:

$$T(z) = \frac{1}{2(k+h_G)}\sum_a (J^aJ^a)(z) \tag{786}$$

Having calculated the OPE $T(z)J^a(w)$ now we turn to the singular terms in the OPE $T(z)T(w)$:

$$\overline{T(z)T(w)} = \tag{787}$$
$$\frac{1}{2(k+h_G)}\frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\sum_a\left[\overline{T(z)J^a(x)}J^a(w) + J^a(x)\overline{T(z)J^a(w)}\right]$$
$$= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

with

$$c = \frac{k\dim g}{k+h_G} \tag{788}$$

In the components we have

$$L_n = \frac{1}{2(k+h_G)} \sum_a \sum_m : J^a_m J^a_{n-m} : \tag{789}$$

Normal ordering is necessary only for $n \neq 0$, since for these n J^a_m and J^a_{n-m} commute. For $n = 0$ normal ordering means us usual that positive indices modes placed at the rightmost position. Collecting all, we have the following set of the commutation relations

$$\begin{aligned} [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\ [L_n, J^a_m] = -mJ^a_{n+m} \\ [J^a_n, J^b_m] = \sum_c if_{abc}J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0} \end{aligned} \tag{790}$$

Proof via the mode expansion

$$\begin{aligned} 2(k+h_G)[L_n, J^a_m] = \sum_b \sum_r [J^b_r J^b_{n-r}, J^a_m] = \\ \sum_b \sum_r J^b_r [J^b_{n-r}, J^a_m] + [J^b_r, J^a_m] J^b_{n-r} \\ \sum_b \sum_r J^b_r \left(\sum_c if_{bac} J^c_{n-r+m} + k(n-r)\delta_{ab}\delta_{n-r+m,0} \right) + \\ \sum_b \sum_r \left(\sum_c if_{bac} J^c_{r+m} + kr\delta_{ab}\delta_{r+m,0} \right) J^b_{n-r} \end{aligned} \tag{791}$$

The delta symbols terms from the both lines yield:

$$-2kmJ^a_{n+m} \tag{792}$$

The first terms in both lines should be brought to the normal ordered form with lower index placed to the left position:

$$\begin{aligned} &\sum_b \sum_r \sum_c if_{bac} J^b_r J^c_{n-r+m} = \\ &\sum_b \sum_{r \leq \frac{n+m}{2}} \sum_c if_{bac} J^b_r J^c_{n-r+m} + \sum_b \sum_{r > \frac{n+m}{2}} \sum_c if_{bac} (J^c_{n-r+m} J^b_r + \sum_d if_{bcd} J^d_{n+m}) = \end{aligned} \tag{793}$$

$$\sum_b \sum_r \sum_c i f_{bac} : J^b_r J^c_{n-r+m} : + \sum_b \sum_{r>\frac{n+m}{2}} \sum_c i f_{bac} \sum_d i f_{bcd} J^d_{n+m} =$$
$$\sum_b \sum_r \sum_c i f_{bac} : J^b_r J^c_{n-r+m} : +2 \sum_{r>\frac{n+m}{2}} h_G J^a_{n+m}$$

$$\sum_b \sum_r \sum_c i f_{bac} J^c_{r+m} J^b_{n-r} = \tag{794}$$
$$\sum_b \sum_{r\leq\frac{n-m}{2}} \sum_c i f_{bac} J^c_{r+m} J^b_{n-r} + \sum_b \sum_{r>\frac{n-m}{2}} \sum_c i f_{bac} (J^b_{n-r} J^c_{r+m} - \sum_d i f_{bcd} J^d_{n+m}) =$$
$$\sum_b \sum_r \sum_c i f_{bac} : J^c_{r+m} J^b_{n-r} : - \sum_b \sum_{r>\frac{n+m}{2}} \sum_c i f_{bac} \sum_d i f_{bcd} J^d_{n+m} =$$
$$\sum_b \sum_r \sum_c i f_{bac} : J^c_{r+m} J^b_{n-r} : -2 \sum_{r>\frac{n-m}{2}} h_G J^a_{n+m}$$

Changing the summing variable $r \to r+m$ the normal ordered terms get canceled due to difference in the order of the b and c indices. The difference of the second terms gives

$$-2m h_G J^a_{n+m} \tag{795}$$

as we expected.

$$[L_n, L_m] = \frac{1}{2(k+h_G)} \sum_a \sum_r [L_n,\ J^a_r J^a_{m-r}] = \tag{796}$$
$$\frac{1}{2(k+h_G)} \sum_a \sum_r \left([L_n,\ J^a_r] J^a_{m-r} + J^a_r [L_n,\ J^a_{m-r}]\right) =$$
$$\frac{1}{2(k+h_G)} \sum_a \sum_r \left(-r J^a_{n+r} J^a_{m-r} + (r-m) J^a_r J^a_{n+m-r}\right)$$

Bringing both terms to the normal ordered form, i.e. moving the higher index to the right position we obtain:

$$[L_n, L_m] = \frac{1}{2(k+h_G)} \sum_a \sum_r \left(-r : J^a_{n+r} J^a_{m-r} : +(r-m) : J^a_r J^a_{n+m-r} :\right) \tag{797}$$
$$+\frac{k\text{dim}g}{2(k+h_G)} \delta_{m+n} \left(\sum_{r>\frac{m-n}{2}} (-r)(n+r) + \sum_{r>\frac{m+n}{2}} (r-m) r \right)$$

Changing the summation variable in the first term $r' = r+n$ the first two term yield

$$(n-m) L_{m+n} \tag{798}$$

The last line can be written as

$$\frac{k\mathrm{dim}g}{2(k+h_G)}\delta_{m+n}\left(\sum_{r>\frac{m-n}{2}}(-r)(n+r)+\sum_{r>\frac{m+n}{2}}(r-m)r\right)= \tag{799}$$
$$\frac{k\mathrm{dim}g}{2(k+h_G)}\delta_{m+n}\sum_{r=0}^{m}r(m-r)=\frac{k\mathrm{dim}g}{12(k+h_G)}(n^3-n)\delta_{m+n}$$

Lecture 20
Representations of the affine algebras

Here we will review the Cartan-Weyl basis of the algebra and the general facts on the highest weight representations.

In the Cartan-Weyl basis the generators are constructed as follows.

One first finds the maximal set of commuting generators: $H^i, i = 1, \ldots r$:

$$[H^i, H^j] = 0 \tag{800}$$

r is called the rank of the algebra. This set of generators form the Cartan subalgebra h.

The generators of the Cartan algebra can all be diagonalized simultaneously.

The remaining generators can be chosen to satisfy:

$$[H^i, E^\alpha] = \alpha^i E^\alpha \tag{801}$$

The vector is called root and E^α is the corresponding ladder operator. Equation (801) via its Hermitian conjugate, shows that if $-\alpha$ is a root as well with

$$E^{-\alpha} = (E^\alpha)^\dagger \tag{802}$$

In the following Δ denotes the set of all roots. To find the remaining commutators we first observe that the Jacobi identity implies:

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta] \tag{803}$$

If $\alpha + \beta \in \Delta$, the commutator $[E^\alpha, E^\beta]$ must be proportional to $E^{\alpha+\beta}$, and it must vanish if $\alpha + \beta$ is not root. When $\alpha = -\beta$, $[E^\alpha, E^{-\alpha}]$ commutes with all H^i, which is possible only if it is a linear combination of the generators of the Cartan subalgebra. The normalization of the ladder operator is fixed by setting this commutator equal to $2\alpha \cdot H/|\alpha|^2$ where

$$\alpha \cdot H = \sum_i^r \alpha^i H^i, \qquad |\alpha|^2 = \sum_i^r \alpha^i \alpha^i \tag{804}$$

So, the set of the commutation relations in the Cartan-Weyl basis is

$$\begin{aligned}
&[H^i, H^j] = 0 \\
&[H^i, E^\alpha] = \alpha^i E^\alpha \\
&[E^\alpha, E^\beta] = N_{\alpha,\beta} E^{\alpha+\beta}, && \text{if } \alpha + \beta \in \Delta \\
&\qquad\quad\;\; = 2\alpha \cdot H/|\alpha|^2 && \text{if } \alpha = -\beta \\
&\qquad\quad\;\; = 0 && \text{otherwise}
\end{aligned} \tag{805}$$

The Killing form

$$K(X,Y) = \frac{1}{2g}\mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y) \qquad (806)$$

g is the dual Coxeter number of the algebra. The standard basis $\{J^c\}$ is understood to be orthonormal with respect to K:

$$K(J^a, J^b) = \delta^{ab} \qquad (807)$$

The same normalization holds for the generators of the Cartan subalgebra:

$$K(H^i, H^j) = \delta^{ij} \qquad (808)$$

The cyclic property of the trace yields the identity:

$$K([Z,X],Y]) + K(X,[Z,Y]) = 0 \qquad (809)$$

Hence we obtain:

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2} \qquad (810)$$

Positive roots

Let us fix the basis in the space of roots $\{\beta_1, \cdots \beta_r\}$. Any root can be expanded in this basis:

$$\alpha = \sum_1^r n_i\beta_i \qquad (811)$$

α is said positive if the first nonzero number in the sequence $(n_1, n_2, \cdots, n_r)$. The set of positive roots we denote by Δ_+.

A simple root α_i is defined to be a root that cannot be written as the sum of two positive roots. There necessarily r simple roots and their set $\{\alpha_1, \cdots \alpha_r\}$ provides the most convenient basis for the r -dimensional space of roots.

Highest root

A distinguished element of Δ is the highest root θ. It is unique root and which, in the expansion $\sum_i m_i\alpha_i$ the sum $\sum_i m_i$ is maximized.

Highest weight representations

For an arbitrary representation a basis $|\lambda\rangle$ can be found such that

$$H^i|\lambda\rangle = \lambda^i|\lambda\rangle \qquad (812)$$

. The eigenvalues λ^i form the vector $\lambda = (\lambda^1, \cdots \lambda^r)$ called a weight. Roots are weights of the adjoint representation. The commutator (801) shows that E^α changes the eigenvalue of a state by α:

$$H^iE^\alpha|\lambda\rangle = [H^iE^\alpha]|\lambda\rangle + E^\alpha H^i|\lambda\rangle = (\alpha^i + \lambda^i)E^\alpha|\lambda\rangle \qquad (813)$$

so that $E^\alpha|\lambda\rangle$ if not zero, must be proportional to a state $|\lambda+\alpha\rangle$. This why E^α is called ladder operator. Let us consider finite-dimensional representations. For these we will get very important relation. For any state in a finite-dimensional representation, there are necessarily two possible integer p and q such that

$$(E^\alpha)^{p+1}|\lambda\rangle \sim E^\alpha|\lambda+p\alpha\rangle = 0 \tag{814}$$

$$(E^{-\alpha})^{q+1}|\lambda\rangle \sim E^{-\alpha}|\lambda-q\alpha\rangle = 0 \tag{815}$$

for any root α. Notice that the triplet of generators E^α, $E^{-\alpha}$, and $\alpha\cdot H/|\alpha|^2$ forms an $su(2)$ subalgebra analogues to the set $[J^+, J^-, J^3]$ with commutation relations

$$[J^+, J^-] = 2J^3, \qquad [J^3, J^\pm] = \pm J^\pm \tag{816}$$

Therefore if $|\lambda\rangle$ belongs to a finite-dimensional representation, its projection onto the $su(2)$ subalgebra associated with the root α must also be finite-dimensional. Let the dimension of the latter be $2j+1$; then from the state $|\lambda\rangle$ the state with highest $J^3 = \alpha\cdot H/|\alpha|^2$ projection ($m=j$) can be reached by a finite number, say p applications of $J^+ = E^\alpha$, whereas, say, q, applications of $J^- = E^{-\alpha}$ lead to the state with $m=-j$

$$j = \frac{(\alpha,\lambda)}{|\alpha|^2} + p \qquad -j = \frac{(\alpha,\lambda)}{|\alpha|^2} - q \tag{817}$$

Eliminating j from the above two equations yields:

$$2\frac{(\alpha,\lambda)}{|\alpha|^2} = -(p-q) \tag{818}$$

Hence any weight in a finite-dimensional representation is such that $2\frac{(\alpha,\lambda)}{|\alpha|^2}$ is an integer.

Since the weights are roots of the adjoint representation the scalar products of the simple roots defines the integer entries Cartan matrix:

$$A_{ij} = \frac{2(\alpha_i\cdot\alpha_j)}{\alpha_j^2} \tag{819}$$

Finding all the Cartan matrices leads to the Dynkin classification of the Lie algebras.

Among all the weights in the representation the highest weight is the one for which the sum of the coefficients expansions in the basis of simple roots is maximal. As a result for any positive root α $\lambda+\alpha$ cannot be a weight, so that

$$E^{\alpha}|\lambda\rangle = 0, \tag{820}$$

for any positive root.

Starting from the highest weight state $\lambda\rangle$,all the states in the representation space can be obtained by the action of the lowering operators as

$$E^{-\beta}E^{-\gamma}\cdots E^{-\eta} \qquad \text{for} \qquad \beta,\gamma,\ldots,\eta \in \Delta_+ \tag{821}$$

Highest weights of the affine algebra

In the Cartan-Weyl basis the commutation relation of the affine algebra takes the form:

$$\begin{aligned}
[H^i_n, H^j_m] &= kn\delta^{ij}\delta_{n+m,0} \\
[H^i_n, E^{\alpha}_m] &= \alpha^i E^{\alpha}_{n+m} \\
[E^{\alpha}_n, E^{\beta}_m] &= N_{\alpha,\beta}E^{\alpha+\beta}_{n+m}, && \text{if } \alpha+\beta\in\Delta \\
&= \frac{2}{|\alpha|^2}\left(\alpha\cdot H_{n+m} + kn\delta_{n+m,0}\right) && \text{if } \alpha = -\beta \\
&= 0 && \text{otherwise}
\end{aligned} \tag{822}$$

The highest weight state now is defined to satisfy:

$$\begin{aligned}
H^i_n|\lambda\rangle = E^{\pm\alpha}_n|\lambda\rangle &= 0, && n>0 \\
H^i_0|\lambda\rangle = \lambda^i|\lambda\rangle, \qquad \text{and} \qquad E^{\alpha}_0|\lambda\rangle &= 0, && \alpha > 0
\end{aligned} \tag{823}$$

Consider again the $su(2)$ subalgebra generated by: E^{α}_0, $E^{-\alpha}_0$, $\frac{2}{|\alpha|^2}\alpha\cdot H_0$. Commutation relations imply:

$$\langle\lambda|E^{\alpha}_0E^{-\alpha}_0|\lambda\rangle = \langle\lambda|[E^{\alpha}_0E^{-\alpha}_0]|\lambda\rangle = \frac{2}{|\alpha|^2}\alpha\cdot\lambda\langle\lambda|\lambda\rangle \geq 0 \tag{824}$$

Hence we must have $\alpha\cdot\lambda \geq 0$.

Now look another $su(2)$ subalgebra generated by: E^{α}_{-1}, $E^{-\alpha}_1$, $\frac{2}{|\alpha|^2}(-\alpha\cdot H_0 + k)$. From (822) we have

$$\langle\lambda|E^{-\alpha}_1E^{\alpha}_{-1}|\lambda\rangle = \langle\lambda|[E^{-\alpha}_1E^{\alpha}_{-1}]|\lambda\rangle = \frac{2}{|\alpha|^2}(-\alpha\cdot\lambda + k)\langle\lambda|\lambda\rangle \geq 0 \tag{825}$$

Restrict ourself for simplicity to the case of unitary algebras for which all roots normalized to 2.

Since the component of the J_3 generatot of $su(2)$ are integer , and we know that for any weight λ $\frac{2}{|\alpha|^2}\alpha \cdot \lambda$ is integer, we obtain that k is integer.

Then it follows from (825) that any highest weight shold satisfy the inequility

$$\alpha \cdot \lambda \leq k \tag{826}$$

The condition (827) is stringent for the highest root θ

$$\theta \cdot \lambda \leq k \tag{827}$$

Using the expression (790) for L_0 we derive the conformal weight of the highest weight state:

$$L_0|\lambda\rangle = \frac{C_\lambda}{2(k+h_G)}|\lambda\rangle \tag{828}$$

where C_λ is the quadratic Casimir of the representation λ.

Let us specialize to the $SU(2)$ group. Note that the normalization of structure constant, $f^{ijk} = \sqrt{2}\epsilon^{ijk}$. It comes from the requirement $\mathrm{Tr}(T^iT^j) = \delta^{ij}$, which implies that for $SU(2)$ we should take $T^i = \frac{\sigma^i}{\sqrt{2}}$. Because of the $\sqrt{2}$ in the commutation rules, we need to take

$$I^{\pm} = \frac{1}{\sqrt{2}}(J_0^1 \pm iJ_0^2) \qquad \text{and} \qquad I^3 = \frac{1}{\sqrt{2}}J_0^3 \tag{829}$$

to give a conventionally normalized $su(2)$ algebra $[I^+, I^-] = 2I^3$, $[I^3, I^{\pm}] = \pm I^{\pm}$, in which $2I^3$ has integer eigenvalue in any finite dimensional representation. But from the commutation relation of the affine $su(2)$ algebra we find that

$$\tilde{I}^+ = \frac{1}{\sqrt{2}}(J_{+1}^1 - iJ_{+1}^2), \quad \tilde{I}^- = \frac{1}{\sqrt{2}}(J_{-1}^1 + iJ_{-1}^2) \quad \text{and} \quad \tilde{I}^3 = \frac{1}{2}k - \frac{1}{\sqrt{2}}J_0^3 \tag{830}$$

as well satisfy $[\tilde{I}^+, \tilde{I}^-] = 2\tilde{I}^3$, $[\tilde{I}^3, \tilde{I}^{\pm}] = \pm\tilde{I}^{\pm}$, so $2\tilde{I}^3 = k - 2I^3$ also has integer eigenvalues. It follows that $k \in \mathbb{Z}$ for unitary highest weight representations. Since the quadratic Casimir in the chosen normalization, in the representation j has value $C_j = 2j(j+1)$, therefore in the adjoint representation $j = 1$, $C_{\text{adj}} = 4$, $h_{SU(2)} = 2$,

and the central charge of the corresponding affine algebra is

$$c = \frac{3k}{k+2} \tag{831}$$

Here we have one root and all weights are given by half-integer numbers j.

$$0 < \langle j|\tilde{I}^+\tilde{I}^-|j\rangle = \langle j|[\tilde{I}^+\tilde{I}^-]|j\rangle = \langle j|k - 2I^3|j\rangle = k - 2j \tag{832}$$

and the highest weights of the $su(2)$ affine algebra are given by the half-integer j satisfying the inequality:

$$2j \leq k \tag{833}$$

The conformal weight of these states are:

$$h_j = \frac{j(j+1)}{k+2} \tag{834}$$

The matrix of the modular transformation is

$$S_{aj} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2a+1)(2j+1)\pi}{k+2}\right) . \tag{835}$$

Characters are

$$\chi_l = \frac{\Theta_{l+1,k+2} - \Theta_{-l-1,k+2}}{\Theta_{1,2} - \Theta_{-1,2}} \tag{836}$$

$$\Theta_{m,k}(\tau, z, u) = e^{-2\pi imu} \sum_{n \in \mathbb{Z}+m/2k} e^{2\pi im(n^2\tau - nz)} \tag{837}$$

Lecture 21: Coset models

GKO construction

Assume we have subgroup H of group G: $H \subset G$. We denote the G currents by J_G^a and the H currents by J_H^i, where i runs only over the ajoint representation of H, i.e. from 1 to $|H| \equiv \dim H$. We can now construct two stress-energy tensors

$$T_G(z) = \frac{1}{2(k_G + h_G)} \sum_{a=1}^{|G|} : J_G^a(z) J_G^a(z) : \tag{838}$$

and also

$$T_H(z) = \frac{1}{2(k_H + h_H)} \sum_{a=1}^{|H|} : J_H^i(z) J_H^i(z) : \tag{839}$$

Now we have:

$$T_G(z) J_H^i(w) \sim \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial J_H^i(w)}{(z-w)} \tag{840}$$

but as well that

$$T_H(z) J_H^i(w) \sim \frac{J_H^i(w)}{(z-w)^2} + \frac{\partial J_H^i(w)}{(z-w)} \tag{841}$$

We see that the OPE of $(T_G - T_H)$ with J_H^i is non-singular. Since T_H above is constructed entirely from H currents J_H^i it also follows that $T_{G/H} \equiv T_G - T_H$ has a nonsingular OPE with all of T_H. This means that

$$T_G = (T_G - T_H) + T_H \equiv T_{G/H} + T_H \tag{842}$$

gives an orthogonal decomposition of the Virasoro algebra generated by T_G into two commuting Virasoro subalgebras, $[T_{G/H}, T_H] = 0$. To compute the central charge of the Virasoro subalgebra generated by $T_{G/H}$, we note that the most singular part of the OPE of two T_G's decomposes as

$$T_G T_G = \frac{c_G/2}{(z-w)^4} \sim T_{G/H} T_{G/H} + T_H T_H \sim \frac{c_G/2 + c_H/2}{(z-w)^4} \tag{843}$$

The result is

$$c_{G/H} = c_G - c_H = \frac{k_G |G|}{k_G + h_G} - \frac{k_H |H|}{k_H + h_H} \tag{844}$$

To understand better the states that arise in the G/H theory, we need to consider how the representation of G decompose under (842). We denote the representation space of affine G at level k_G by $|c_G, \lambda_G\rangle$, where c_G is the central charge appropriate to k_G, and λ_G is the highest weight of the vacuum representation. Under the orthogonal decomposition of the Virasoro algebra $T_G = T_{G/H} + T_H$, this space must decompose as some direct sum of irreducible representations

$$|c_G, \lambda_G\rangle = \oplus_j |c_{G/H}, h^j_{G/H}\rangle \otimes |c_H, \lambda^j_H\rangle \tag{845}$$

where $|c_{G/H}, h^j_{G/H}\rangle$ denotes an irreducible representation of $T_{G/H}$ with lowest L_0 eigenvalue $h^j_{G/H}$. It follows immediately from the decomposition (845) that the character of an affine G representation with highest weight λ^a satisfies:

$$\chi^{k_G}_{\lambda^a_G}(\tau) = \sum_j \chi^{c_{G/H}}_{h_{G/H}(\lambda^a_G,\lambda^j_H)}(\tau)\chi^{k_H}_{\lambda^j_H} \equiv \chi_{G/H} \cdot \chi^H_{\lambda_H} \tag{846}$$

In (846) the L_0 eigenvalues $h_{G/H}$ characterizing the $T_{G/H}$ Virasoro representation depend implicitly on the highest weights λ^a_G and λ^j_H characterizing the associated G and H representation. On the r.h.s. of (846) we have introduced a matrix notation.

Under modular transformation:

$$\gamma : \tau \to \frac{a\tau + b}{c\tau + d} \tag{847}$$

the characters allowed at any given fixed level k_G of an affine algebra transform as a unitary representation

$$\chi^{k_G}(\tau') = M^{k_G}(\gamma)\chi^{k_G}(\tau) \tag{848}$$

with $(M^{k_G})^b_a$ a unitary matrix. But from (846) we also have

$$\chi^{k_G}(\tau') = \chi^{k_{G/H}}(\tau')M^{k_H}(\gamma)\chi^{k_H}(\tau) \tag{849}$$

Linear independence of the G and H characters then allows us to solve for the modular transformation properties of the $T_{G/H}$ characters as

$$\chi^{k_{G/H}}(\tau') = M^{k_G}(\gamma)\chi^{k_{G/H}}(\tau)M^{k_H}(\gamma)^{-1} \tag{850}$$

By other words

$$M^{G/H}(\gamma)_{(\lambda^a_G,\lambda^i_H);(\lambda^b_G,\lambda^j_H)} = M^{k_G}(\gamma)_{\lambda^a_G,\lambda^b_G} M^{k_H}(\gamma)_{\lambda^i_H,\lambda^j_H} \tag{851}$$

Also we have

$$h_{G/H}(\lambda^a_G, \lambda^i_H) = h^G_{\lambda^a_G} - h^H_{\lambda^i_H} + n \tag{852}$$

Lagrangian of coset model: Gauged WZW model

Let G be a compact, simply connected, non-abelian group. The G/H coset CFT, where H is a subgroup of G, can be described in terms of a gauged WZW action, where the symmetry

$$g \to hgh^{-1} \tag{853}$$

$g \in G$, $h \in H$ is gauged away. An H Lie algebra valued world sheet vector field A is added to the system, and the G/H action on a world-sheet without boundary becomes,

$$\begin{aligned} S^{G/H} &= S^{G/H} + S^{\text{gauge}} \\ &= \frac{k_G}{4\pi}\left[\int_\Sigma d^2 z L^{\text{kin}} + \int_B \omega^{WZW}\right] \\ &+ \frac{k_G}{2\pi}\int_\Sigma d^2 z \text{Tr}[A_{\bar{z}}\partial_z g g^{-1} - A_z \partial_{\bar{z}} g g^{-1} + A_{\bar{z}} g A_z g^{-1} - A_{\bar{z}} A_z] \end{aligned} \tag{854}$$

Introduce H group element valued world sheet fields U and $\tilde{U}$ as

$$A_z = \partial_z U U^{-1} \tag{855}$$

$$A_{\bar{z}} = \partial_{\bar{z}} \tilde{U} \tilde{U}^{-1} \tag{856}$$

Then the coset action becomes:

$$S^{G/H} = S^{G/H}(U^{-1} g \tilde{U}) - S^H(U^{-1}\tilde{U}) \tag{857}$$

The level k_H of the S^H term is related to k_G through the embedding index of H in G. The model has then the following symmetries . First of all one should identify configurations related by the local gauge transformation

$$\begin{aligned} g(z,\bar{z}) &\to h(z,\bar{z}) g(z,\bar{z}) h^{-1}(z,\bar{z}) \\ U(z,\bar{z}) &\to h(z,\bar{z}) U(z,\bar{z}) \\ \tilde{U}(z,\bar{z}) &\to h(z,\bar{z}) \tilde{U}(z,\bar{z}) \end{aligned} \tag{858}$$

with $h(z,\bar{z}) \in H$.

Minimal model

Now we turn to the specific case of coset spaces of the form $G \times G/G$, where the group G in the denominator is the diagonal subgroup. If we call the generators of the two groups in the numerator $J^a = J^a_{(1)} + J^a_{(2)}$.The most singular part of their OPE is

$$J^a(z)J^b(w) \sim J^a_{(1)}(z)J^b_{(1)}(w) + J^a_{(2)}(z)J^b_{(2)}(w) \sim \frac{(k_1+k_2)\delta^{ab}}{(z-w)^2} + \cdots \tag{859}$$

so that the level the G in the denominator is determined by the diagonal embedding to be $k_1 + k_2$. A simple example of this type is provided by

$$G/H = SU(2)_k \times SU(2)_1 \times /SU(2)_{k+1} \tag{860}$$

in which case

$$c_{G/H} = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+1+2} = 1 - \frac{6}{(k+2)(k+3)} \tag{861}$$

We can write the conformal dimensions of the minimal models in the form:

$$h_{r,s} = \frac{(r-s)^2}{4} + \frac{r^2-1}{4(k+2)} - \frac{s^2-1}{4(k+3)} \tag{862}$$

Solving $r^2 - 1 = 4j(j+1)$, and $s^2 - 1 = 4n(n+1)$ we get $r = 2j+1$, and $s = 2n+1$.

The parafermion $\mathcal{A}^{PF(k)} = \frac{SU(2)_k}{U(1)_k}$

The chiral algebra of this theory has a set of irreducible representations described by pairs (j,n) where $j \in \frac{1}{2}Z$, $0 \le j \le k/2$, and n is an integer defined modulo $2k$. The pairs are subject to a constraint $2j+n = 0\text{mod}2$, and an equivalence relation $(j,n) \sim (k/2-j, k+n)$. The character of the representation (j,n), denoted by $\chi_{j,n}(q)$, is determined implicitly by the decomposition

$$\chi_j^{SU(2)}(q) = \sum_{n=-k}^{k+1} \chi_{j,n}^k(q)\psi_n(q)\,. \tag{863}$$

The action of modular group on the character is

$$\chi_{j,n}^k(q') = \sum_{(j',n')} S^{PF}_{(j,n),(j'n')}\chi^k_{j',n'}(q) \tag{864}$$

and the PF S-matrix is

$$S^{PF}_{(j,n),(j'n')} = \frac{1}{\sqrt{2k}} e^{\frac{i\pi nn'}{k}} S_{jj'}\,, \tag{865}$$

where $S_{jj'}$ defined in (1076).

When combining left and right-movers, the simplest modular invariant partition function of the parafermion theory is obtained by summing over all distinct representations

$$Z = \sum_{(j,n)\in PF_k} |\chi_{j,n}|^2\,. \tag{866}$$

The parafermion theory has a global Z_k symmetry under which the fields $\psi_{j,n}$ generating the representation (j,n) transform as

$$g: \quad \psi_{j,n} \to \omega^n \psi_{j,n}, \quad \omega = e^{\frac{2\pi i}{k}} . \tag{867}$$

Therefore we can orbifold the theory by this group. Taking the symmetric orbifold by Z_k of (866) leads to the partition function

$$Z = \sum_{(j,n)\in PF_k} \chi_{j,n} \bar{\chi}_{j,-n} . \tag{868}$$

We see that effect of the orbifold is to change the relative sign between the left and right movers of the $U(1)$ group with which we orbifold. Therefore the Z_k orbifold of the parafermion theory at level k is T-dual to the original theory. This fact will be the basis of many constructions in the main text.

$$\begin{aligned} J^+(z) &= \sqrt{k}\psi_{\rm par} e^{i\sqrt{2/k}\phi(z)} \\ J^-(z) &= \sqrt{k}\psi^\dagger_{\rm par} e^{-i\sqrt{2/k}\phi(z)} \\ J^0(z) &= \sqrt{2k}\partial_z\phi(z) \end{aligned} \tag{869}$$

Lecture 22: C=1 Orbifold model

In conformal field theory the notion of orbifold acquires the following meaning. We start by taking a given modular invariant theory $\mathcal{T}$, whose Hilbert space admits a discrete symmetry G consistent with the interactions or operator algebra of the theory, and constructing a modded-out theory $\mathcal{T}/G$ that is also modular invariant.

Orbifold conformal field theories occasionally have a geometric interpretation as σ-models whose target space is the geometrical orbifold. This we shall confirm momentarily in the case of the S^1/Z_2 example. But we shall see examples however where the geometrical interpretation non-existent. Consequently it is preferable to regard orbifold conformal field theories from the more abstract standpoint of modding out a modular invariant theory by a Hilbert space symmetry. We will consider here the case of the abelian symmetry group G.

The construction of an orbifold conformal field theory $\mathcal{T}/G$ begins with a Hilbert space projection onto G invariant states.

Therefore the first part of the partition function has the form:

$$Z^{(1)}_{\text{orb}} = |q|^{-c/12}\frac{1}{|G|}\text{Tr}\sum_{g\in G} g q^{L_0}\bar{q}^{\bar{L}_0} \tag{870}$$

This means that we sum over all insertions of the operator realization of group element g in the trace over states, or alternatively this can be understood as twisting in the time direction. To have modular invariant partition function we should add contribution of the configurations twisted in the space direction $x(z+1) = h\dot{x}(z)$:

$$Z_{\text{orb}} = |q|^{-c/12}\frac{1}{|G|}\sum_{g,h\in G}\text{Tr}_h g q^{L_0}\bar{q}^{\bar{L}_0} \tag{871}$$

Now let us consider the free boson orbifolded by Z_2 symmetry $\phi\to-\phi$.

According to (871) the partition function takes the form

$$Z_{\text{orb}} = |q|^{-1/12}\frac{1}{2}\text{Tr}_+(1+G)q^{L_0}\bar{q}^{\bar{L}_0} + |q|^{-1/12}\frac{1}{2}\text{Tr}_-(1+G)q^{L_0}\bar{q}^{\bar{L}_0} \tag{872}$$

Here G is the operator realization of the inversion $\phi\to-\phi$, Tr_+ denotes the trace over untwisted sector considered before and Tr_- denotes the trace over twisted anti-periodic sector $\phi(x+L,t) = -\phi$.

Let us analyze the chiral contributions. Consider the projected contribution in untwisted sector:

$$f_{0,\frac{1}{2}} = \mathrm{Tr}_+ G q^{L_0 - 1/24} = \mathrm{Tr}_+ G q^{\sum_{n\in\mathbb{N}} a_n a_n - 1/24} \tag{873}$$

To compute this trace note that the inversion flips signs all the creation and annihilation operators and the winding and momentum zero modes. Therefore we can split the untwisted chiral Hilbert space into $G = \pm 1$ eigenspaces $\mathcal{H}^{\pm}$:

$$\begin{aligned} \mathcal{H}^+ = &\{\alpha_{-n_1} \cdots \alpha_{-n_{2k}}(|m,n\rangle + |-m,-n\rangle)\} + \\ &\{\alpha_{-n_1} \cdots \alpha_{-n_{2k+1}}(|m,n\rangle - |-m,-n\rangle)\} \end{aligned} \tag{874}$$

$$\begin{aligned} \mathcal{H}^- = &\{\alpha_{-n_1} \cdots \alpha_{-n_{2k}}(|m,n\rangle - |-m,-n\rangle)\} + \\ &\{\alpha_{-n_1} \cdots \alpha_{-n_{2k+1}}(|m,n\rangle + |-m,-n\rangle)\} \end{aligned} \tag{875}$$

Since the first line in $\mathcal{H}^+$ has the same L_0 eigenvalue as the first line in $\mathcal{H}^-$, but opposite G eigenvalue, their contributions get canceled. By the same reason get canceled also contribution of the second lines in $\mathcal{H}^+$ and $\mathcal{H}^-$. Hence the only contribution comes from the $|0,0\rangle$ sector. Summarizing we obtain:

$$f_{0,\frac{1}{2}} = q^{-1/24} \prod_{n=1}^{\infty} \frac{1}{1+q^n} = \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}} \tag{876}$$

Now we address the twisted sectors. Oscillators in the twisted sectors half-integer modded. Therefore the L_0 has an expansion:

$$L_0 = \frac{1}{2} \sum_{n\in\mathbb{Z}+1/2} : a_n a_n := \sum_{n\in\mathbb{N}+1/2} a_n a_n + \frac{1}{2} \sum_{n\in\mathbb{N}+1/2} n \tag{877}$$

Using ζ-function regularized value of the sum of half-integer numbers, we get:

$$L_0 = \sum_{n\in\mathbb{N}+1/2} a_n a_n + \frac{1}{48} \tag{878}$$

Hence we have for chiral contributions:

$$f_{\frac{1}{2},0} = \mathrm{Tr}_- q^{L_0} = q^{1/48} \prod_{r=\mathbb{N}+1/2} \frac{1}{1-q^r} = \sqrt{\frac{\eta(\tau)}{\theta_4(\tau)}} \tag{879}$$

$$f_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}_- G q^{L_0} = q^{1/48} \prod_{r=\mathbb{N}+1/2} \frac{1}{1+q^r} = \sqrt{\frac{\eta(\tau)}{\theta_3(\tau)}} \tag{880}$$

Also taking into account that we have two twisted sectors differing by two ways of acting on ϕ: $\phi \to -\phi$ and $\phi \to 2\pi - \phi$ we end up with

$$Z_{\rm orb} = \frac{1}{2}\left(Z(R) + |f_{0,\frac{1}{2}}|^2 + 2|f_{\frac{1}{2},0}|^2 + 2|f_{\frac{1}{2},\frac{1}{2}}|^2\right) = \\ \frac{1}{2}Z(R) + \frac{|\eta|}{|\theta_2|} + \frac{|\eta|}{|\theta_3|} + + \frac{|\eta|}{|\theta_4|} \tag{881}$$

Lecture 23
Boundary c=1 systems

Let us consider a conformal field theory on the $\sigma - \tau$ strip, $0 \le \sigma \le \pi$, periodic in the τ-direction with a period T. The manifold is an annulus with the modular parameter $q \equiv \exp(-2\pi i T)$. Given certain boundary conditions on the boundaries of the annulus, labelled α and β, the partition function is:

$$Z_{\alpha\beta} = \mathrm{Tr}\exp(-2\pi i T H_{\alpha\beta}) \ , \tag{882}$$

where $H_{\alpha\beta}$ is the Hamiltonian corresponding to these boundary conditions. This is the open-string channel. One may also calculate the partition function using the Hamiltonian acting in the σ-direction . This will be the Hamiltonian $H^{(P)}$ for the cylinder, which is related by the exponential mapping $\zeta = \exp(-i(t+i\sigma))$ to the Virasoro generators in the whole ζ-plane by $H^{(P)} = L_0^{(P)} + \overline{L_0}^{(P)} - c/12$, where we have used the superscript to stress that they are not the same as the generators of the boundary Virasoro algebra. To every boundary condition α, there corresponds a particular boundary state $|\alpha\rangle$ in the Hilbert space of the closed strings; this enables us to compute the partition function by the following formula:

$$Z_{\alpha\beta} = \langle\alpha|\exp(-\pi i H^{(P)}/T)|\beta\rangle = \langle\alpha|(\tilde{q}^{1/2})^{L_0^{(P)}+\overline{L_0}^{(P)}-c/12}|\beta\rangle \ , \tag{883}$$

where $\tilde{q} \equiv e^{-2\pi i/T}$.

This is the closed-string tree channel.

The boundary entropy for each boundary is defined by Affleck and Ludwig:

$$g_\alpha = \langle 0|\alpha\rangle \ . \tag{884}$$

The phases of $|0\rangle$ and $|\alpha\rangle$ can be chosen such that $\langle 0|\alpha\rangle$ is real and positive for all boundary states $|\alpha\rangle$. In the path integrand language, g_α is the value of the disc diagramm satisfying α type boundary condition. Affleck and Ludwig have shown that, at least in conformal perturbation theory, the value of g always decreases with the flow of the renormalization group.

The equality of (882) and (883) provides a convenient way to calculate g, as shown in the following.

Boundary action

$$S = \int \partial X \bar{\partial} X dx^+ dx^- \tag{885}$$

The variation of the action in the presence of the boundary takes form:

$$\delta S = -\int 2\partial\bar{\partial} X \delta X dx^+ dx^- + \int (\partial X dx^+ - \bar{\partial} X dx^-)\delta X \tag{886}$$

Now let us take:

$$x^+ = \tau + \sigma \tag{887}$$

$$x^- = \tau - \sigma \tag{888}$$

$$\partial_\tau = \partial + \bar{\partial} \tag{889}$$

$$\partial_\sigma = \partial - \bar{\partial} \tag{890}$$

Assume that boundary located at $\sigma = 0$ (open string loop channel).

In this case the boundary term takes the form:

$$\int (\partial X dx^+ - \bar{\partial} X dx^-)\delta X = \int \partial_\sigma X \delta X d\tau \tag{891}$$

and we have two kinds of boundary condition: Neumann boundary condition

$$\partial_\sigma X|_{\sigma=0} \tag{892}$$

and Dirichlet boundary condition is

$$X|_{\sigma=0} = 0 \tag{893}$$

If the boundary located at $\tau = 0$, the boundary term takes the form:

$$\int (\partial X dx^+ - \bar{\partial} X dx^-)\delta X = \int \partial_\tau X \delta X d\sigma \tag{894}$$

and the Neumann boundary condition takes the form:

$$\partial_\tau X|_{\tau=0} \tag{895}$$

and the Dirichlet boundary condition is

$$X|_{\tau=0} = 0 \tag{896}$$

Neumann boundary conditions

Closed string tree-channel

The action with the Neumann boundary condition can include also the Wilson line term at the boundary :

$$S = \frac{1}{2\pi}\int_0^\pi d\sigma \int d\tau \partial_\alpha X \partial^\alpha X + \sum_B \frac{iy_B}{\pi}\int_B dX \ , \tag{897}$$

where B labels boundaries and y_B are the constant modes of the $U(1)$ gauge potential coupling to the boundaries (and are periodic, with periods π/R). We assume that the boundaries carry also Chan-Paton factors whose index we choose to take two values, 1 and 2. Thus at the enhanced symmetry point we have a $U(2)$ gauge symmetry, which is generically broken down to $U(1)\times U(1)$ by the Wilson line.

Here we consider a world-sheet with two boundaries, the annulus diagram.

In order to find the boundary entropy, the theory should be compared in two channels: the closed-string tree channel and the open-string loop channel.

In the closed-string channel the first task is to find the boundary states $|N_i\rangle$, with Chan-Paton factor i, which are found by imposing the corresponding boundary conditions. The boundary is located at $\tau = 0$ and one has the usual condition of vanishing momentum flow:

$$\partial_\tau X(\sigma, 0) = P(\sigma, 0) = 0 \ . \tag{898}$$

Inserting the mode expansion:

$$X(\sigma,\tau) = x + 2wR\sigma + \frac{p\tau}{R} + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}[\alpha_n e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n e^{-2in(\tau+\sigma)}] \ , \tag{899}$$

where p and w are correspondingly integer momenta and winding numbers, we get:

$$p = 0, \qquad \alpha_n = -\tilde{\alpha}_{-n} \ . \tag{900}$$

Taking into account the properties of coherent state and the $U(1)$ modes y_i we get for $|N_i\rangle$:

$$|N_i\rangle = g_N \sum_w e^{-2iy_i wR} \exp\left(\sum_{n>0} -\frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right)|0, w\rangle \ , \tag{901}$$

where the phase factor comes from the Wilson line term:

$$\frac{iy_B}{\pi}\int d\sigma \partial_\sigma X = 2iy_B w_B R \ , \tag{902}$$

where w_B is the winding number of the boundary.

We see that the normalization factor $g_N = \langle 0|N_i\rangle$ gives us the boundary entropy. Inserting the expression for $|N_i\rangle$ and the closed string Hamiltonian

$$H = \frac{p^2}{4R^2} + w^2R^2 + N + \tilde{N} - \frac{1}{12} \tag{903}$$

in (883), we obtain for the partition function in the closed string channel:

$$Z_{12} = g_N^2 \langle N_2| \exp\left(\frac{-i\pi w^2 R^2}{T}\right) \exp\left[\frac{-i\pi}{T}(N + \tilde{N} - \frac{1}{12})\right] |N_1\rangle =$$
$$\frac{g_N^2}{\eta(\tilde{q})} \sum_w e^{-2i(y_1-y_2)wR} \exp\left(\frac{-i\pi w^2 R^2}{T}\right) = \frac{g_N^2}{\eta(\tilde{q})}\theta_3\left(\frac{-R^2}{T}, (y_2 - y_1)R\right) \ , \tag{904}$$

where

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{905}$$

is the Dedekind function, and

$$\theta_3(\tau, z) = \sum_{n=-\infty}^{\infty} \exp(i\pi n^2\tau + 2inz) \tag{906}$$

is the third theta function with the modular parameter τ. To calculate g_{N_i} one turns to the open string loop channel.

Open string loop-channel

The Hamiltonian should be computed with a given boundary condition. First consider the mode expansion for X. The action takes form

$$S = \frac{1}{2\pi}\int_0^{\pi} d\sigma \int d\tau \partial_\alpha X \partial^\alpha X + \frac{y_2}{\pi}\int d\tau \partial_\tau X - \frac{y_1}{\pi}\int d\tau \partial_\tau X \ , \tag{907}$$

We see that the space-time momentum gets modified

$$P = \int_0^{\pi}\left(\frac{1}{\pi}\partial_\tau X + \frac{y_2}{\pi}\delta(\sigma) - \frac{y_1}{\pi}\delta(\pi - \sigma)\right) = \frac{p}{R} + \frac{y_2 - y_1}{\pi} \tag{908}$$

Therefore the mode expansion of the solution of the equation of motion with the Wilson lines parameters y_1 and y_2 is:

$$X = x + \left(\frac{p}{R} + \frac{y_2 - y_1}{\pi}\right)\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n \cos(n\sigma)\exp(-in\tau) \ , \tag{909}$$

where p is an integer. Inserting this in the open-string Hamiltonian, we obtain:

$$H = \frac{1}{2}\left(\frac{p}{R} + \frac{y_2 - y_1}{\pi}\right)^2 + N - \frac{1}{12} \ . \tag{910}$$

The partition function in this channel is :

$$Z = \frac{e^{\frac{-iT(y_2-y_1)^2}{\pi}}}{\eta(q)} \sum_p \exp\left(\frac{-i\pi T p^2}{R^2} - \frac{2iTp(y_2-y_1)}{R}\right) \tag{911}$$
$$= e^{\frac{-iT(y_2-y_1)^2}{\pi}} \frac{1}{\eta(q)} \theta_3\left(-\frac{T}{R^2}, -\frac{T(y_2-y_1)}{R}\right) .$$

Equating (904) and (911) and using the properties of modular transformations:

$$\theta_3\left(\frac{1}{\tau}, z\right) = \tau^{1/2} e^{i\tau z^2/\pi} \theta_3(\tau, \tau z) \tag{912}$$

$$\eta(\tilde{q}) = (-T)^{1/2} \eta(q) , \tag{913}$$

we obtain:

$$g_N^2 = R , \tag{914}$$

Dirichlet boundary conditions

Closed string tree-channel

The boundary entropy for the open string with Dirichlet boundary condition is similarly evaluated, starting again with the closed string channel.

The boundary condition determining the boundary state is :

$$X|_{\tau=0} = y \tag{915}$$

leading to:

$$w = 0, \qquad \alpha_n = \tilde{\alpha}_{-n} . \tag{916}$$

From these conditions, for the boundary state located at the point y we get

$$|D_y\rangle = g_{D_y} \delta(x-y) \exp\left(\sum_{n>0} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right)|0\rangle = g_{D_y} \sum_p e^{\frac{-ipy}{R}} \exp\left(\sum_{n>0} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right)\left|\frac{p}{R}, 0\right\rangle . \tag{917}$$

Inserting this in (883) we have for the partition function in this channel:

$$Z_{12} = g_D^2 \langle D_{y_2}| \exp\left(\frac{-i\pi p^2}{4R^2 T}\right) \exp\left[\frac{-i\pi}{T}(N + \tilde{N} - \frac{1}{12})\right] |D_{y_1}\rangle =$$
$$\frac{g_D^2}{\eta(\tilde{q})} \sum_p e^{\frac{-ip(y_1-y_2)}{R}} \exp\left(\frac{-i\pi p^2}{4R^2 T}\right) = \frac{g_D^2}{\eta(\tilde{q})} \theta_3\left(-\frac{1}{4TR^2}, \frac{y_2-y_1}{2R}\right) . \tag{918}$$

Open string loop channel

In order to analyse the open-string loop channel, according to (882), the Hamiltonian must be expressed with the Dirichlet boundary condition. The mode expansion of the coordinate X with the boundary conditions

$$X|_{\sigma=0} = y_1 \tag{919}$$

$$X|_{\sigma=\pi} = y_2 \tag{920}$$

is

$$X = y_1 + \left(\frac{y_2 - y_1}{\pi} + 2wR\right)\sigma + i\sum_{n\neq 0}\frac{1}{n}\alpha_n \sin(n\sigma)\exp(-in\tau) \tag{921}$$

Substituting it in the open-string Hamiltonian leads to:

$$H = \frac{1}{2}\left(\frac{y_2 - y_1}{\pi} + 2wR\right)^2 + N - \frac{1}{12}\ . \tag{922}$$

Finally, the partition function in this channel is:

$$Z_{12} = \frac{e^{-iT(y_2-y_1)^2}}{\eta(q)}\sum_w \exp\left(-4i\pi T w^2 R^2 - 4iwRT(y_2 - y_1)\right) = \frac{1}{\eta(q)}\theta_3(-4TR^2, -2RT(y_2 - y_1))\ . \tag{923}$$

Equating the partition functions in the two channels and using (912), one obtains:

$$g_D^2 = \frac{1}{2R}\ . \tag{924}$$

Results are consistent with T-duality.

Neumann-Dirichlet mixed annulus diagram

Closed string tree-channel

$$Z_{ND} = g_D g_N \langle D|\exp\left[\frac{-i\pi}{T}(N + \tilde{N} - \frac{1}{12})\right]|N\rangle = \frac{1}{\sqrt{2}}\tilde{q}^{-1/24}\prod_1^\infty \frac{1}{1+\tilde{q}^n} = \sqrt{\frac{\eta(\tilde{q})}{\theta_2(\tilde{q})}} \tag{925}$$

Open string loop channel

The mode expansion with the Dirichlet boundary conditions on one side

$$\partial_\tau X|_{\sigma=0} = 0 \tag{926}$$

and Neumann boundary condition on other side

$$\partial_\sigma X|_{\sigma=\pi} = 0 \tag{927}$$

is

$$X^{DN} = x_0 + 2i \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{a_n}{n} e^{-in\tau} \sin(n\sigma) \tag{928}$$

The partition function is

$$Z_{ND} = q^{1/48} \prod_{n\in\mathbb{N}+\frac{1}{2}} \frac{1}{1-q^n} = \sqrt{\frac{\eta(q)}{\theta_4(q)}} \tag{929}$$

Lecture 24
Boundary rational conformal field theory

Suppose we have a theory containing besides tensor energy-momentum T the set of conserved currents $W^{(r)}$. The boundary conditions in the upper half plane are

$$T(z) = \bar{T}(\bar{z})|_{z=\bar{z}} \qquad W^{(r)}(z) = \overline{W^{(r)}}(\bar{z})|_{z=\bar{z}} \tag{930}$$

The first of these conditions has the direct physical meaning of the absence of energy-momentum flow across the boundary $T_{xt} = 0$. In this case, the eigenstates of $H_{\alpha\beta}$ will be organized into highest weight representations R_i of the extended algebra. These representations R_i will be labelled by an index i whose specification includes the L_0-eigenvalue of the highest weight state. We then define the non-negative integer $n^i_{\alpha\beta}$ to be the number of times that the representation i occurs in the spectrum of $H_{\alpha\beta}$. The partition function in the open string channel (882) is then

$$Z_{\alpha\beta} = \operatorname{Tr}\exp(-2\pi i T H_{\alpha\beta}) = \sum_i n^i_{\alpha\beta}\chi_i(q) \tag{931}$$

where $\chi_i(q)$ is the character of the representation i.

Corresponding boundary states should satisfy

$$\left(W^{(r)}_n - (-)^{h_W}\overline{W}^{(r)}_{-n}\right)|\alpha\rangle = 0 \tag{932}$$

Define the anti-unitary operator U acting in the way

$$U\overline{W}^{(r)}_{-n} = (-)^{h_W}\overline{W}^{(r)}_{-n}U \tag{933}$$

Using $|j, N\rangle$, $N \in \mathbb{N}$, to denote an orthonormal basis of R_i, one can define Ishibashi states:

$$|j\rangle\rangle = \sum_{N=0}^{\infty} |j, N\rangle \otimes U\overline{|j, N\rangle} \tag{934}$$

Let us show that the Ishibashi states are solutions of (932). To see this, consider the vectors $\langle k, N_1| \otimes U\overline{\langle l, N_2|}$. Then

$$\langle k, N_1| \otimes U\overline{\langle l, N_2|}\left(W^{(r)}_n - (-)^{h_W}\overline{W}^{(r)}_{-n}\right)|j\rangle\rangle \tag{935}$$

$$\sum_{N=0}^{\infty}\langle k, N_1| \otimes U\overline{\langle l, N_2|}\left(W^{(r)}_n - (-)^{h_W}\overline{W}^{(r)}_{-n}\right)|j, N\rangle \otimes U\overline{|j, N\rangle}$$

$$= \sum_{N=0}^{\infty}\left[\langle k, N_1|(W^{(r)}_n|j, N\rangle\overline{\langle l, N_2||j, N\rangle}^* - (-)^{h_W}\langle k, N_1||j, N\rangle\overline{\langle l, N_2|U^\dagger\overline{W}^{(r)}_{-n}U\overline{|j, N\rangle}}^*\right]$$

$$= \langle k, N_1|(W^{(r)}_n|l, N_2\rangle - \overline{\langle l, N_2|\overline{W}^{(r)}_{-n}\overline{|k, N_1\rangle}}^* = 0$$

Ishibashi states satisfy

$$\langle\langle j|\tilde{q}^{L_0-c/24}|i\rangle\rangle = \delta_{i,j}\chi_i(\tilde{q}) \tag{936}$$

The boundary states are linear combinations of the Ishibashi states:

$$|\alpha\rangle = \sum B^i_\alpha|i\rangle\rangle \tag{937}$$

Inserting expansions (937) in the expression (883) for the partition function in the closed string channel we obtain:

$$Z_{\alpha\beta} = \sum_i (B^i_\alpha)^* B^i_\beta \chi_i(\tilde{q}) \tag{938}$$

Performing modular transformation we get for partition function in the open string channel:

$$Z_{\alpha\beta} = \sum_{i,j} (B^i_\alpha)^* B^i_\beta S_{ij}\chi_j(q) \tag{939}$$

Equating (931) and (939) we derive

$$\sum_i (B^i_\alpha)^* B^i_\beta S_{ij} = n^j_{\alpha\beta} \tag{940}$$

Cardy has found a solution with the help of the Verlinde formula

$$N^k_{ij} = \sum_l \frac{S_{il}S_{jl}S^*_{lk}}{S_{0l}} \tag{941}$$

In Cardy's solution, the boundary states $|\alpha\rangle$ carry the same labels as the irreducible representations, and their expansion into Ishibashi states is

$$|\alpha\rangle = \sum_i \frac{S_{\alpha i}}{\sqrt{S_{0i}}}|i\rangle\rangle \tag{942}$$

The second part of the condition (930) may be generalized to incorporate a possible "gluing automorphism" Ω

$$W(z) = \Omega\overline{W}(\bar{z})|_{z=\bar{z}} \tag{943}$$

The corresponding boundary state $|\alpha\rangle_\Omega$ satisfy the condition

$$\left(W^{(r)}_n - (-)^{h_W}\Omega(\overline{W}^{(r)}_{-n})\right)|\alpha\rangle_\Omega = 0 \tag{944}$$

The state $|\alpha\rangle_\Omega$ is given by a linear combination of twisted Ishibashi states $|i\rangle\rangle_\Omega$:

$$|i\rangle\rangle_\Omega = (\mathrm{Id}\otimes V_\Omega)|i\rangle\rangle \tag{945}$$

where V_Ω is the representation of Ω on Hilbert space.

Lecture 25
Cardy-Lewellen equation

Let us derive the cluster condition for usual branes. Consider a boundary state

$$|\alpha\rangle = \sum_i B^i_\alpha |i\rangle\rangle \tag{946}$$

where i runs over primaries, and $|i\rangle\rangle$ are Ishibashi states. Recall the relation between coefficients B^i_α and one-point functions

$$\langle \Phi_{(i\bar{i})}(z,\bar{z})\rangle_\alpha = \frac{U^i_\alpha \delta_{i^*\bar{i}}}{(z-\bar{z})^{2\Delta_i}} \tag{947}$$

in the presence of the boundary condition α:

$$U^i_\alpha = \frac{B^i_\alpha}{B^0_\alpha} e^{i\pi\Delta_i} \tag{948}$$

Consider now two-point function $\langle \Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2)\rangle_\alpha$ in the presence of boundary in two pictures. In the first picture one applies first bulk OPE

$$\Phi(z_1,\bar{z}_1)_{(i\bar{i})}\Phi(z_2,\bar{z}_2)_{(j\bar{j})} = \sum_{k,\bar{k}} \frac{C^{(k\bar{k})}_{(i\bar{i})(j\bar{j})}}{(z_1-z_2)^{\Delta_i+\Delta_j-\Delta_k}(\bar{z}_1-\bar{z}_2)^{\Delta_{\bar{i}}+\Delta_{\bar{j}}-\Delta_{\bar{k}}}}\Phi_{(k\bar{k})}(z_2,\bar{z}_2)+\dots \tag{949}$$

and then evaluates one-point function resulting in:

$$\langle \Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2)\rangle_\alpha = \sum_{k,a,\bar{a}} C^{(k,k^*)}_{(i\bar{i})(j\bar{j})} U^k_\alpha \mathcal{F}_{k^*}\begin{bmatrix} j & \bar{i} \\ i^* & \bar{j}\end{bmatrix} \tag{950}$$

where

$$\mathcal{F}_{k^*}\begin{bmatrix} j & \bar{i} \\ i^* & \bar{j}\end{bmatrix} \tag{951}$$

is conformal block.

In the second picture one first applies bulk-boundary OPE

$$\Phi_{(i\bar{i})}(z,\bar{z}) = \sum_{m,s} \frac{R^{(i\bar{i})}_{m,s,(\alpha)}}{(z-\bar{z})^{\Delta_i+\Delta_{\bar{i}}-\Delta_m}}\psi^{\alpha\alpha,s}_m + \dots \tag{952}$$

where, and index s counts different boundary fields and runs $s=1,\dots n^m_{\alpha\alpha}$, where $n^m_{\alpha\alpha}$ coefficient of character χ_m in the annulus partition function between brane

α with itself, and then evaluates two-point function of boundary fields resulting in

$$\langle\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2)\rangle_\alpha = \sum_{m,s_1,s_2} R^{(i\bar{i})}_{m,s_1(\alpha)} R^{(j\bar{j})}_{m^*,s_2(\alpha)} c_m^{\alpha,s_1,s_2} \mathcal{F}_{m^*}\left[\begin{array}{cc} \bar{i} & j \\ i^* & \bar{j} \end{array}\right] \tag{953}$$

where

$$\langle\psi_m^{\alpha\alpha,s_1}(x_1)\psi_n^{\alpha\alpha,s_2}(x_2)\rangle = \frac{c_m^{\alpha,s_1,s_2}\delta_{mn^*}}{|x_2-x_1|^{2\Delta_m}} \tag{954}$$

and

$$\mathcal{F}_{m^*}\left[\begin{array}{cc} \bar{i} & j \\ i^* & \bar{j} \end{array}\right] \tag{955}$$

is conformal block. Using braiding relations between chiral blocks

$$\mathcal{F}_{k^*}\left[\begin{array}{cc} j & \bar{i} \\ i^* & \bar{j} \end{array}\right] = \sum_m B^{(+)}_{k^*m^*}\left[\begin{array}{cc} j & \bar{i} \\ i^* & \bar{j} \end{array}\right] \mathcal{F}_{m^*}\left[\begin{array}{cc} \bar{i} & j \\ i^* & \bar{j} \end{array}\right] \tag{956}$$

one derives:

$$\sum_k C^{(k,k^*)}_{(i\bar{i})(j\bar{j})} U^k_\alpha B^{(+)}_{k^*m^*}\left[\begin{array}{cc} j & \bar{i} \\ i^* & \bar{j} \end{array}\right] = \sum_{s_1,s_2} R^{(i\bar{i})}_{m,s_1,(\alpha)} R^{(j\bar{j})}_{m^*,s_2,(\alpha)} c_m^{\alpha,s_1,s_2} \tag{957}$$

Putting $m=0$ one obtains:

$$\sum_k C^{(k,k^*)}_{(ii^*)(jj^*)} U^k_\alpha B^{(+)}_{k^*0}\left[\begin{array}{cc} j & i^* \\ i^* & j^* \end{array}\right] = U^i_{(\alpha)} U^j_{(\alpha)} \tag{958}$$

where we took into account that $R^{i\bar{i}}_{0(\alpha)} = U^i_\alpha \delta_{i^*\bar{i}}$. The traditionally used reflection amplitudes differ by phase

$$U^i_{(\alpha)} = \tilde{U}^i_{(\alpha)} e^{i\pi\Delta_i} \tag{959}$$

They have the advantage, that related to boundary states coefficients without phase factor:

$$\tilde{U}^i_{(\alpha)} = \frac{B^i_\alpha}{B^0_\alpha} \tag{960}$$

Recalling relation between braiding and fusion matrices:

$$B^{(+)}_{pq}\left[\begin{array}{cc} i & j \\ k & l \end{array}\right] = e^{i\pi(\Delta_k+\Delta_l-\Delta_p-\Delta_q)} F_{pq}\left[\begin{array}{cc} i & l \\ k & j \end{array}\right] \tag{961}$$

and symmetry properties of fusion matrix

$$F_{pq}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right] = F_{p^*q^*}\left[\begin{array}{cc} l & i^* \\ j^* & k \end{array}\right] \tag{962}$$

we receive that $\tilde{U}^i_{(\alpha)}$ obey the equation:

$$\sum_k C^{(k,k^*)}_{(ii^*)(jj^*)} \tilde{U}^k_\alpha F_{k0} \begin{bmatrix} i^* & i \\ j & j \end{bmatrix} = \tilde{U}^i_{(\alpha)} \tilde{U}^j_{(\alpha)} \tag{963}$$

Using the relation

$$C^p_{ij} = \frac{\xi_i \xi_j}{\xi_0 \xi_p} \frac{1}{F_{p,0} \begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}, \tag{964}$$

where

$$\xi_i = \sqrt{C_{ii^*} F_i}\,. \tag{965}$$

we obtain

$$\sum_k \tilde{U}^k N^k_{ij} \frac{\xi_i \xi_j}{\xi_0 \xi_k} = \tilde{U}^i \tilde{U}^j\,, \tag{966}$$

where N^k_{ij} are the fusion coefficients. Defining

$$\tilde{U}^k = \Psi^k \frac{\xi_k}{\xi_0} \tag{967}$$

one can write (966) in the form:

$$\sum_k \Psi^k N^k_{ij} = \Psi^i \Psi^j\,. \tag{968}$$

In rational conformal field theory one has also the relation

$$F_k = \frac{S_{00}}{S_{0k}}\,, \tag{969}$$

where S_{ab} is the matrix of the modular transformations.

In RCFT two-points functions can be normalized to 1. Therefore in RCFT $\xi_k = \frac{\sqrt{S_{00}}}{\sqrt{S_{0k}}}$. Eq. (968) is solved by

$$\Psi^k_a = \frac{S_{ak}}{S_{0a}}\,. \tag{970}$$

Taking into account the relation between one-point functions U^k and coefficients of the boundary state B^k

$$\tilde{U}^k = \frac{B^k}{B^0}\,, \tag{971}$$

we obtain the formulae for the Cardy states:

$$B^k_a = \frac{S_{ak}}{\sqrt{S_{0k}}}\,, \tag{972}$$

Lecture 26
Boundary WZW model

Let us consider boundary conditions satisfying the relations:

$$J^a = \bar{J}^a, \qquad a = 1, \cdots \dim G \tag{973}$$

As we explained before in the absence of the boundary the WZW action possesses the affine $G_L \times G_R$ symmetry:

$$g \to h_L(z) g h_R^{-1}(\bar{z}) \tag{974}$$

The boundary condition (973) implies that the symmetry (974) is broken to the diagonal symmetry, requiring that $h_L = h_R = h$ on the boundary. The presence of this symmetry constraints the boundary conditions that can be placed on g. Allowing g(boundary) $= f$ for some $f \in G$ we must also allow g(boundary) $= hfh^{-1} = \mathcal{C}_f$ for every $h \in G$. This means that g on the boundary takes value in the conjugacy class containing f. Now we are going to write down the corresponding boundary Lagrangian. Recall that to write the WZW model we used the three-manifold B satisfying the condition $\partial B = \Sigma$. When the world-sheet Σ has itself boundaries, it cannot be the boundary of a three dimensional manifold, since a boundary cannot have boundary. To define the WZW term for this case, one should fill holes in the worldsheet by adding auxiliary discs, and extend the mapping from the worldsheet into the group manifold to these discs. One further demands that the whole disc D is mapped into a region inside the conjugacy class in which the corresponding boundary lies. B will then be defined as a three-manifold bounded by the union $\Sigma \cup D$,which now has no boundaries. To make the action independent on the location of the auxiliary disc inside conjugacy class we should demand that

$$\omega^{\mathrm{WZW}}(g)|_{g \in \mathcal{C}_f} = d\omega_f \tag{975}$$

and modify the action by the boundary term

$$S^{\mathrm{boundary}} = S^{\mathrm{WZW}} - \frac{k}{4\pi} \int_D \omega_f \tag{976}$$

First of all using the Polyakov-Wiegmann identities it is easy to check that indeed (975) for $\mathcal{C}_f = kfk^{-1}$ fulfilled with:

$$\omega_f(k) = \mathrm{Tr}(k^{-1}dkfk^{-1}dkf^{-1}) \tag{977}$$

Now we can check that the action (976) is invariant under the transformation

$$g \to h_L(z) g h_R^{-1}(\bar{z}) \tag{978}$$

with the boundary condition $h_L(z)|_{\text{boundary}} = h_R(\bar{z})_{\text{boundary}} = h(\tau)$.

Under this transformation, the change in the L^{kin} term is canceled by the corresponding Σ integral of the boundary term from the change in the ω^{WZW} term. In the presence of a world-sheet boundary there remains the contribution from D to the latter change. And since according to the Polyakov-Wiegmann identity

$$\omega^{\text{WZW}}(hgh^{-1}) - \omega^{\text{WZW}}(g) = d(\text{Tr}[h^{-1}dh(gh^{-1}dhg^{-1} - g^{-1}dg - dgg^{-1})]) \tag{979}$$

we have

$$\Delta(S^{\text{kin}} + S^{\text{WZW}}) = \frac{k}{4\pi} \int_D \text{Tr}[h^{-1}dh(gh^{-1}dhg^{-1} - g^{-1}dg - dgg^{-1})] \tag{980}$$

On the other hand under this transformation $k \to hk$ and

$$\omega_f(hk) - \omega_f(k) = \text{Tr}[h^{-1}dh(gh^{-1}dhg^{-1} - g^{-1}dg - dgg^{-1})] \tag{981}$$

Here $g = \mathcal{C}_f = kfk^{-1}$.

Equations (980) and (981) imply invariance of the action (976) under (978).

Let us now elaborate boundary equation of motion. The full derivatives terms from (731) gives the following contribution to the boundary terms:

$$\int \text{Tr}[\delta g g^{-1} \partial_z g g^{-1} dz - \delta g g^{-1} \partial_{\bar{z}} g g^{-1} d\bar{z}] \tag{982}$$

To find contribution from the ω^{WZW} and $\omega_f(k)$ terms note the identity:

$$\text{Tr}(g^{-1}\delta g(g^{-1}dg)^2)|_{g=C} - \delta\omega_f(k) = dA_f(k)\,. \tag{983}$$

$$A_f(k) = \text{Tr}[k^{-1}\delta k(f^{-1}k^{-1}dkf - fk^{-1}dkf^{-1})]\,. \tag{984}$$

Using the parametrization

$$z = \tau + i\sigma \qquad \bar{z} = \tau - i\sigma \tag{985}$$

and taking boundary at the $\sigma = 0$ we get

$$\int \text{Tr}\Big[\delta g g^{-1}\partial_z g g^{-1} - \delta g g^{-1}\partial_{\bar{z}} g g^{-1} + k^{-1}\delta k f^{-1}k^{-1}\frac{dk}{d\tau}f - k^{-1}\delta k f k^{-1}\frac{dk}{d\tau}f^{-1}\Big] d\tau \tag{986}$$

Remembering that $g = kfk^{-1}$, after some transformation we obtain:

$$\int \mathrm{Tr}\Big[2\delta k k^{-1}\left(g^{-1}\partial_{\bar z} g + \partial_z g g^{-1}\right)\Big] d\tau \tag{987}$$

Therefore boundary equations of motion imply

$$g^{-1}\partial_{\bar z} g + \partial_z g g^{-1} = 0 \tag{988}$$

or recalling the definition of currents

$$J = \bar J \tag{989}$$

as expected.

Global issues

The modified action (976) is independent, by construction, of continuous deformation of D inside $\mathcal{C}_f$. However, in general, the second homotopy of a conjugacy class $\pi_2(\mathcal{C}_f)$ is non-trivial. If we compare then the value of the action for D and D', two different choices of embedding the disc in $\mathcal{C}_f$ with the same boundary, D' may not be a continuous deformation of D in $\mathcal{C}_f$. In that case the above analysis does not imply that the two ways to evaluate the action (976) agree. Since there is no natural way to choose between the two embeddings, (976) is not yet a well defined action. In particular, for $G = SU(2)$ the conjugacy clases $\mathcal{C}_f$ have the topology of S^2, the two-sphere generated by all possible axes of rotation by a fixed angle in three dimensions. One may then choose D and D' such that their union covers the whole of S^2. In that case the difference between the action S_D, the value of (976) with embedding D, and $S_{D'}$ with embedding D' is

$$\Delta S = \frac{k}{4\pi}\Big[\int_B \omega^{\mathrm{WZW}} - \int_{\mathcal{C}_f}\omega^f\Big] \tag{990}$$

where B is the three-volume in $SU(2)$ bounded by the two-sphere $\mathcal{C}_f$. For the case of $SU(2)$, which has the topology of S^3, the form ω^{WZW} 4 times the volume form on the unit three sphere. For $\mathcal{C}_f$ with $f = e^{i\psi\sigma_3}$, the first term in (990) is

$$\int_B \omega^{\mathrm{WZW}} = 8\pi(\psi - \frac{1}{2}\sin(2\psi)) \tag{991}$$

As to the two-form ω^f it is proportional to the volume form of the unit two-sphere. We can directly compute for $\mathcal{C}_\psi$

$$\omega^f = \sin(2\psi)\mathrm{vol}S^2 \tag{992}$$

This gives for the change in the action for two topologically different embeddings

$$\Delta S = 2k\psi \tag{993}$$

Although this is non-zero, the quantum theory is still well-defined if ΔS is an multiple of 2π. We find that the possible conjugacy classes on which a boundary state live are quantized, the corresponding ψ must satisfy

$$\psi = 2\pi \frac{j}{k} \tag{994}$$

Boundary states geometry

Given a boundary state,

$$|a\rangle_C = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} |j\rangle\rangle \tag{995}$$

the shape of the brane can be deduced by considering the overlap of the boundary state with the localised bulk state $|\vec{\theta}\rangle$, with $\vec{\theta}$ denoting the three $SU(2)$ angles in some coordinate system. As we will see, the boundary state wave function over the configuration space of all localised bulk states peaks precisely at those states which are localised at positions derived by the effective methods in the previous sections. In the large k limit, the eigen-position bulk state is given by

$$|\vec{\theta}\rangle = \sum_{j,m,m'} \sqrt{2j+1} \mathcal{D}^j_{mm'}(\vec{\theta}) |j,m,m'\rangle\,, \tag{996}$$

where $\mathcal{D}^j_{mm'}$ are the Wigner $\mathcal{D}$-functions:

$$\mathcal{D}^j_{mm'} = \langle jm|g(\vec{\theta})|jm'\rangle, \quad \langle jm|jm'\rangle = \delta_{m,m'} \tag{997}$$

where $|jm\rangle$ are a basis for the spin j representation of $SU(2)$. To calculate the overlap with the boundary state, we will need the knowledge of S-matrix of $SU(2)$ at level k,

$$S_{aj} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(2a+1)(2j+1)\pi}{k+2}\right)\,. \tag{998}$$

In the large-k limit the ratio of S-matrix elements appearing in the boundary state simplifies to

$$\frac{S_{aj}}{\sqrt{S_{0j}}} \sim \frac{(2(k+2))^{1/4}}{\sqrt{\pi(2j+1)}} \sin[(2j+1)\hat{\psi}]\,, \tag{999}$$

where, to shorten the notation, we have introduced $\hat{\psi} = \frac{(2a+1)\pi}{k+2}$. Using these results, the overlap between the boundary state and the localised bulk state becomes

$$\langle\vec{\theta}|a\rangle_C \sim \sum_{j,m,n} \frac{(2(k+2))^{1/4}}{\sqrt{\pi}} \sin[(2j+1)\hat{\psi}]\mathcal{D}^j_{mm}(g(\vec{\theta})\,. \qquad (1000)$$

Finally, one needs the property of the Wigner D-functions that $\sum_n \mathcal{D}^j_{nn}(g) = \frac{\sin(2j+1)\psi}{\sin\psi}$, where ψ is the angle of the standard metric (756) and defined by the relation $\mathrm{Tr}g = 2\cos\psi$. The overlap (1078) becomes

$$\langle\vec{\theta}|a\rangle_C \sim \frac{(2k+4)^{1/4}}{\sqrt{\pi}\sin\psi} \sum_j \sin[(2j+1)\hat{\psi}]\sin[(2j+1)\psi] \qquad (1001)$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle\vec{\theta}|a\rangle_C \sim \frac{\sqrt{\pi}(k+2)^{1/4}}{2^{7/4}\sin\psi}\delta(\psi-\hat{\psi})\,. \qquad (1002)$$

Hence we see that the brane wave function is localized on $\psi = \text{const.}$.

Lecture 27
Non-maximally symmetric boundary states in WZW model

Lagrangian construction Let us consider the D-brane as a product of the conjugacy class with the $U(1)$ subgroup:

$$g|_{\text{boundary}} = LC = Lhfh^{-1} \tag{1003}$$

where $L \in U(1)$. We should check that on this subset exists a two-form $\omega^{(2)}$ satisfying the condition:

$$d\omega^{(2)} = \omega^{\text{WZW}}|_{\text{boundary}} \tag{1004}$$

It may be easily checked using the Polyakov-Wiegmann identity:

$$\omega^{\text{WZW}}(LC) = \omega^{\text{WZW}}(L) + \omega^{\text{WZW}}(C) - d\,\text{Tr}(L^{-1}dLdCC^{-1}) \tag{1005}$$

Using that for the abelian group, L, $\omega^{\text{WZW}}(L) = 0$, and

$$\omega^{\text{WZW}}(C) = d\omega^{f}(h) = d\,\text{Tr}h^{-1}dhfh^{-1}dhf^{-1} \tag{1006}$$

we get that indeed

$$\omega^{\text{WZW}}|_{\text{boundary}} = d\omega^{(2)}(L,h) \tag{1007}$$

where

$$\omega^{(2)}(L,h) = \omega^{f}(h) - \text{Tr}(L^{-1}dLdCC^{-1}) \tag{1008}$$

Now the action is

$$S = S^{\text{WZW}} - \frac{k}{4\pi}\int_D \omega^{(2)}(L,h) \tag{1009}$$

Let us show that the action (1009) is invariant under the symmetry

$$g(z,\bar z) \to h_L(z) g(z,\bar z) h_R(\bar z) \tag{1010}$$

with $h_L(z)|_{\text{boundary}} = h_R(\bar z)_{\text{boundary}} = k(\tau)$, $k \in U(1)$. Under this transformation $L \to kLk$ and $C \to k^{-1}Ck$ and $h \to k^{-1}h$.

Under the transformation (1010), as before the change in the L^{kin} term is canceled by the corresponding Σ integral of the boundary term from the change in the ω^{WZW} term. In the presence of a world sheet boundary there remains the contribution from D to the latter change

$$\Delta S^{\text{WZW}} = \frac{k}{4\pi}\int_D \text{Tr}[k^{-1}dk(g^{-1}dg - gk^{-1}dkg^{-1} - dgg^{-1})] \tag{1011}$$

where $g = LC$. Substituting this value in (1011) we get

$$\Delta S^{\rm WZW} = \frac{k}{4\pi}\int_D {\rm Tr}[k^{-1}dk(C^{-1}dC - Ck^{-1}dkC^{-1} + C^{-1}L^{-1}dLC - dLL^{-1} - dCC^{-1})] \tag{1012}$$

Now we compute $\omega^{(2)}(kLk, k^{-1}h) - \omega^{(2)}(L,h)$ using that

$$\omega^{(f)}(k^{-1}h) - \omega^{(f)}(h) = {\rm Tr}[k^{-1}dk(Ck^{-1}dkC^{-1} + C^{-1}dC + dCC^{-1})] \tag{1013}$$

and

$$\begin{aligned} &{\rm Tr}[(kLk)^{-1}d(kLk)d(k^{-1}Ck)k^{-1}C^{-1}k - L^{-1}dLdCC^{-1}] \\ &= {\rm Tr}[k^{-1}dk(2dCC^{-1} + 2Ck^{-1}dkC^{-1} + L^{-1}dL - C^{-1}L^{-1}dLC)] \end{aligned} \tag{1014}$$

resulting in

$$\begin{aligned} &\omega^{(2)}(kLk, k^{-1}h) - \omega^{(2)}(L,h) = {\rm Tr}[k^{-1}dk(C^{-1}dC - Ck^{-1}dkC^{-1} - \\ &dCC^{-1} - L^{-1}dL + C^{-1}L^{-1}dLC)] \end{aligned} \tag{1015}$$

which cancels (1012).

Geometry

Brane is given by the conjugacy class multiplied by the $U(1)_{\sigma_3}$ group: $\hat{g} \equiv g = hfh^{-1}e^{i\alpha\frac{\sigma_3}{2}} \equiv \mathcal{C}L$. The geometry of the image can be determined as follows . Using the fact that ${\rm Tr}\,\mathcal{C} = {\rm Tr} f = {\rm const} = 2\cos\psi$ we can write

$${\rm Tr}\left(ge^{-i\alpha\frac{\sigma_3}{2}}\right) = 2\cos\psi\,. \tag{1016}$$

From here we see that the element g belongs to the image of the brane surface if and only if there is a $U(1)$ element ($e^{i\alpha\frac{\sigma_3}{2}}$) such that the equation (1016) is satisfied. So let us determine for which g this equation admits solutions for α. Denoting with θ, $\tilde{\phi}$ and ϕ the coordinates of g in the parametrization given in (753), the equation (1016) takes the form

$$\cos\theta\cos(\tilde{\phi} - \frac{\alpha}{2}) = \cos\psi\,, \tag{1017}$$

or equivalently,

$$0 \le \cos^2(\tilde{\phi} - \frac{\alpha}{2}) = \frac{\cos^2\psi}{\cos^2\theta} \le 1\,. \tag{1018}$$

Hence, equation (1018) can be solved for α only when $\cos^2\theta \ge \cos^2\psi$, or equivalently when

$$\cos\tilde{\theta} \ge \cos 2\psi\ , \quad \tilde{\theta} = 2\theta\,. \tag{1019}$$

We see that the image of the brane is a three-dimensional surface defined by the inequality (1019).

Boundary state

Let us start by reviewing the T-duality between a Lens space and the $SU(2)$ theory. Geometrically, a Lens space is obtained by quotienting the group manifold by the right action of the subgroup Z_k of the $U(1)$, and in the Euler coordinates it corresponds to the identification $\varphi \sim \varphi + \frac{4\pi}{k}$. In terms of the $SU(2)$ WZW model this is the orbifold $SU(2)/Z_k^R$, where Z_k^R is embedded in the right $U(1)$. The partition function for this theory is

$$Z = \sum_j \chi_j^{SU(2)}(q)\chi_{jn}^{PF}(\bar{q})\psi_{-n}^{U(1)}(\bar{q}) \tag{1020}$$

and coincides with the one for the $SU(2)$ group, up to T-duality. This relation enables one to construct new D-branes in the $SU(2)$ theory starting from the known ones. As a first step one constructs the brane in the Lens theory. As is usual for orbifolds, this is achieved by summing over images of D-branes under the right Z_k multiplications. Performing then the T-duality on the Lens theory brings us back to the $SU(2)$ theory and maps the orbifolded brane to a new $SU(2)$ brane.

Our starting point is a maximally symmetric A-brane, preserving the symmetries. If we shift the brane by the right multiplication with some element $\omega^l = e^{\frac{2\pi l i}{k}\sigma_3}$ of the Z_k^R group, then the symmetries preserved by this brane are

$$J^a + \omega^l \bar{J}^a \omega^{-l} = 0\,, \quad (a = 1, 2, 3)\,, \tag{1021}$$

while the brane is described by the Cardy state with rotated Ishibashi state

$$|A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{1022}$$

Summing over the images one obtains a Z_k^R invariant state, present in the Lens theory

$$\sum_{l=0}^{k} |A, a\rangle_C^{\omega^l} = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}} \sum_{l=0}^{k} \sum_N |j, N\rangle \otimes (\omega^l \overline{|j, N\rangle})\,. \tag{1023}$$

To compute the sum of the Ishibashi states on the right-hand side, one next uses the orbifold decomposition of $SU(2)_k$

$$SU(2)_k = (\mathcal{A}^{PF(k)} \otimes U(1)_k)/Z_k\,. \tag{1024}$$

This decomposition implies that Ishibashi states for the maximally symmetric A-brane can be written as

$$|A,j\rangle\rangle^{SU(2)} = \sum_{n=1}^{2k} \frac{1+(-1)^{2j+n}}{2} |A,j,n\rangle\rangle_u^{PF} \otimes |A,n\rangle\rangle_u^{U(1)}\,, \tag{1025}$$

where

$$|A,j,n\rangle\rangle_u^{PF} = \sum_N |j,n,N\rangle \otimes \overline{|j,n,N\rangle}\,, \tag{1026}$$

and

$$|Ar\rangle\rangle_u^{U(1)} = \exp\left[\sum_{n=1}^{\infty} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right] \sum_{l\in Z} |\frac{r+2kl}{\sqrt{2k}}\rangle \otimes \overline{|\frac{r+2kl}{\sqrt{2k}}\rangle}\,, \tag{1027}$$

are the A-type Ishibashi states for the parafermion and $U(1)_k$ theories. If the Z_k^R subgroup lies in the $U(1)$ group appearing in the decomposition (1024), then under the action of element $\omega^l \in Z_k^R$ the expression (1025) transform as

$$|A,j\rangle\rangle^{SU(2)} \to \sum_{n=1}^{2k} \frac{1+(-1)^{2j+n}}{2} \omega^{ln} |A,j,n\rangle\rangle_u^{PF} \otimes |A,n\rangle\rangle_u^{U(1)}\,. \tag{1028}$$

Hence summing over images projects onto the Z_k^R-invariant Ishibashi states for which n is restricted to the two values 0 and k. Performing T-duality, flips the sign of the right moving $U(1)$ sector and one gets a B-type Ishibashi state of the original $SU(2)$ theory,

$$|B,j\rangle\rangle^{SU(2)} = \left[\frac{1+(-1)^{2j}}{2}|A,j,0\rangle\rangle_u^{PF} \otimes |B,0\rangle\rangle_u^{U(1)} + \right. \tag{1029}$$
$$\left. \frac{1+(-1)^{2j+k}}{2}|A,j,k\rangle\rangle_u^{PF} \otimes |B,k\rangle\rangle_u^{U(1)}\right]\,,$$

where

$$|Br\rangle\rangle_u^{U(1)} = \exp\left[-\sum_{n=1}^{\infty} \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n}\right] \sum_{l\in Z} |\frac{r+2kl}{\sqrt{2k}}\rangle \otimes \overline{|-\frac{r+2kl}{\sqrt{2k}}\rangle}\,, \tag{1030}$$

is a B-type Ishibashi state of $U(1)_k$ theory satisfying the Neumann boundary conditions. Knowing the T-dual expression of the (1023) allows one to write down the boundary state for the B-type brane

$$|B,a\rangle_C^{SU(2)} = \sum_{j\in Z} \frac{\sqrt{k}S_{aj}}{\sqrt{S_{0j}}} |Aj,0\rangle\rangle_u^{PF} \otimes (|B0\rangle\rangle_u^{U(1)} + \eta |Bk\rangle\rangle_u^{U(1)})\,. \tag{1031}$$

where $\eta = (-1)^{2a}$. In deriving this expression one uses the field identification rule $(j, n) \sim (k/2 - j, k + n)$ and the following property of the matrix of modular transformation (1076)

$$S_{a,k/2-j} = (-1)^{2a} S_{aj} \,. \tag{1032}$$

To derive the symmetries preserved by the B-brane, one observes from (1021) that a Z_k^R invariant superposition of the A-branes preserves only the current $J^3 + \bar{J}^3$ and breaks all other currents; namely, any two Z_k^R images only have this preserved current in common. Performing further T-duality in the $\bar{J}^3$ direction flips the relative sign between the two terms in this current and hence implies that the only current preserved by the B-brane is

$$J^3 - \bar{J}^3 = 0 \,. \tag{1033}$$

Overlap of the state and the coordinate wave function

We will now show that the boundary state (1031) reproduces the effective brane geometry (1019). In the large k limit the second term in (1031) can be ignored. As in the case of Cardy state one should compute the overlap $\langle \vec{\theta} | B, a \rangle_C^{SU(2)}$. We will again use the formula (996), but taking into account that the matrix $\mathcal{D}$ has left and right indices 0. Therefore, the overlap is again given by formula (1078), but with n set to zero. Hence we arrive at the equation

$$\langle \vec{\theta} | B, a \rangle_C^{SU(2)} \sim \sum_j \frac{k^{3/2}}{\pi} \sin[(2j+1)\hat{\psi}] \, \mathcal{D}^j_{00}(g(\vec{\theta})) \tag{1034}$$

Next we will need the relation between the Wigner D-functions and the Legendre polynomials $P_j(\cos\tilde{\theta})$ given by $\mathcal{D}^j_{00} = P_j(\cos\tilde{\theta})$, as well as the formula for the generating function for Legendre polynomials

$$\sum_n t^n P_n(x) = \frac{1}{\sqrt{1 - 2tx + t^2}} \,. \tag{1035}$$

Using these expressions equation (1034) can be simplified to

$$\langle \vec{\theta} | B, a \rangle_C^{SU(2)} \sim \frac{\Theta(\cos\tilde{\theta} - \cos 2\psi)}{\sqrt{\cos\tilde{\theta} - \cos 2\psi}} \,, \tag{1036}$$

where Θ is the step function. This indeed coincides with the expression for the effective geometry (1019).

Open strings in gauged WZW model

As we explained before the action of the gauged WZW model using the Polyakov-Wiegamnn identities can be written in the form:

$$S^{G/H} = S^{G/H}(U^{-1}g\tilde{U}) - S^{H}(U^{-1}\tilde{U}) \tag{1037}$$

Consider the action (1037) on a world-sheet with a boundary. Following the corresponding discussion of the WZW model on a a world-sheet with a boundary we suggest the following boundary conditions:

$$U^{-1}g\tilde{U}|_{\text{boundary}} = (U^{-1}n)f(U^{-1}n)^{-1}, \qquad n, f \in G \tag{1038}$$

and

$$U^{-1}\tilde{U}|_{\text{boundary}} = (U^{-1}p)l^{-1}(U^{-1}p)^{-1} \qquad p, l \in H \tag{1039}$$

Conditions (1038) and (1039) imply

$$g|_{\text{boundary}} = nfn^{-1}plp^{-1} = c_1c_2 \tag{1040}$$

where $c_1 = nfn^{-1}$ and $c_2 = plp^{-1}$, and also on the boundary

$$\tilde{U} = pl^{-1}p^{-1}U \tag{1041}$$

Now we can write the action of the gauged WZW model in the presence of a boundary:

$$S^{G/H} = S^{G/H}(U^{-1}g\tilde{U}) - S^{H}(U^{-1}\tilde{U}) - \frac{k}{4\pi}\int_D \omega^{(f)}(U^{-1}n) + \frac{k}{4\pi}\int_D \omega^{(l^{-1})}(U^{-1}p) \tag{1042}$$

Using again PW identities we obtain

$$\begin{aligned} S^{G/H} &= S^{G/H} + S^{\text{gauge}} \\ &= \frac{k_G}{4\pi}\left[\int_\Sigma d^2z L^{\text{kin}} + \int_B \omega^{WZW}\right] \\ &+ \frac{k_G}{2\pi}\int_\Sigma d^2z \text{Tr}[A_{\bar{z}}\partial_z g g^{-1} - A_z\partial_{\bar{z}} g g^{-1} + A_{\bar{z}} g A_z g^{-1} - A_{\bar{z}}A_z] - \frac{k}{4\pi}\int_D \Omega \end{aligned} \tag{1043}$$

with

$$\begin{aligned} \Omega &= \omega^{(f)}(U^{-1}n) - \omega^{(l^{-1})}(U^{-1}p) \\ &+ \text{Tr}\Big[g^{-1}dgd\tilde{U}\tilde{U}^{-1} - dUU^{-1}dgg^{-1} - dUU^{-1}gd\tilde{U}\tilde{U}^{-1}g^{-1} + dUU^{-1}d\tilde{U}\tilde{U}^{-1}\Big] \end{aligned} \tag{1044}$$

After some straightforward calculations we obtain for boundary term

$$\Omega = \omega^{(f)}(n) + \omega^{(l)}(p) + \text{Tr}(dc_2c_2^{-1}c_1^{-1}dc_1) \tag{1045}$$

It is easy to check that:

$$\omega^{\text{WZW}}(c_1c_2) = d\Omega \tag{1046}$$

Lecture 28
Defects in WZW model

The construction of defects lines is analogous to that of boundary condition. We define defect lines as operators X, satisfying relations:

$$T^{(1)} = T^{(2)} \qquad W^{(1)} = W^{(2)} \tag{1047}$$

$$\bar{T}^{(1)} = \bar{T}^{(2)} \qquad \bar{W}^{(1)} = \bar{W}^{(2)} \tag{1048}$$

After modular transformation these defects are given by operators X, satisfying relations:

$$[L_n, X] = [\bar{L}_n, X] = 0 \tag{1049}$$

$$[W_n, X] = [\bar{W}_n, X] = 0 \tag{1050}$$

As in the case of the boundary conditions, there are also consistency conditions, analogous to the Cardy and Cardy-Lewellen constraints, which must be satisfied by the operator X. For simplicity we shall write all the formulae for diagonal models. To formulate these conditions, one first note that as consequence of (1049) and (1050) X is a sum of projectors

$$X = \sum_{i,\bar{i}} \mathcal{D}^{(i,\bar{i})} P^{(i,\bar{i})} \tag{1051}$$

where

$$P^{(i,\bar{i})} = \sum_{N,\bar{N}} (|i,N\rangle \otimes |\bar{i},\bar{N}\rangle)(\langle i,N| \otimes \langle \bar{i},\bar{N}|) \tag{1052}$$

An analogue of the Cardy condition for defects requires that partition function with insertion of a pair defects after modular transformation can be expressed as sum of characters with non-negative integers.

Namely using a Hamiltonian picture with time moving perpendicular to the lines, the torus partition function may be written

$$Z_{ab} = \mathrm{Tr}\left(X_a^{\dagger} X_b \tilde{q}^{L_0 - \frac{c}{24}} \tilde{\bar{q}}^{\bar{L}_0 - \frac{c}{24}}\right) = \sum_{i,\bar{i}} (\mathcal{D}_a^{(i,\bar{i})})^* \mathcal{D}_b^{(i,\bar{i})} \chi_i(\tilde{q}) \chi_{\bar{i}}(\tilde{\bar{q}}) \tag{1053}$$

A second representation of the same partition function may be obtained by considering time running parallel to the defect lines. In this case, the definition of the disorder line (1050) insures one may still construct two sets of generators W_n

and $\bar{W}_n$ satisfying the chiral algebra. Hence the Hilbert space decomposes into irreducible representations

$$\mathcal{H}_{ab} = \oplus_{i,\bar{i}} V^b_{i,\bar{i};a} \mathcal{R}_i \otimes \bar{\mathcal{R}}_{\bar{i}} \tag{1054}$$

for some non-negative integers $V^b_{i,\bar{i};a}$, and the partition function becomes

$$Z_{ab} = \mathrm{Tr}_{\mathcal{H}_{ab}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = \sum_{i,\bar{i}} V^b_{i,\bar{i};a} \chi_i(q) \chi_{\bar{i}}(\bar{q}) \tag{1055}$$

We may equate these two expressions using the modular transformation properties of the characters

$$V^b_{i,\bar{i};a} = \sum_{j,\bar{j}} S_{ji} S_{\bar{j}\bar{i}} (\mathcal{D}_a^{(i,\bar{i})})^* \mathcal{D}_b^{(i,\bar{i})} \tag{1056}$$

It is found that for diagonal models one can solve this condition taking for each primary a

$$\mathcal{D}_a^{(i,\bar{i})} = \frac{S_{ai}}{S_{0i}} \tag{1057}$$

for which one has:

$$Z_{ab} = \mathrm{Tr}\left(X_a^\dagger X_b \tilde{q}^{L_0 - \frac{c}{24}} \tilde{\bar{q}}^{\bar{L}_0 - \frac{c}{24}} \right) = \sum_{k,i\bar{i}} N^a_{bk} N^k_{i\bar{i}} \chi_i(q) \chi_{\bar{i}}(\bar{q}) \tag{1058}$$

Topological defects can act on boundary states producing new boundary states. The action of defects (1057) on Cardy states is easily obtained using the Verlinde formula:

$$X_a |b\rangle = \sum_d N^d_{ab} |d\rangle \tag{1059}$$

Topological defects can be fused. For defects (1057) again using the Verlinde formula one derives:

$$X_a X_b = \sum_c N^c_{ab} X_c \tag{1060}$$

Lagrangian approach to defects in the WZW model

Let us assume that one has a defect line S separating the world-sheet into two regions Σ_1 and Σ_2. In such a situation the WZW model is defined by pair of maps g_1 and g_2. On the defect line itself one has to impose conditions that relate the two maps. The necessary data are captured by the geometrical structure of a bibrane: a bibrane is in particular a submanifold of the Cartesian product of the group G with itself : $Q \subset G \times G$. The pair of maps (g_1, g_2) are restricted by the requirement that the combined map

$$S \to (G \times G) : s \to (g_1(s), g_2(s) \in Q \tag{1061}$$

takes its value in the submanifold Q. Additionally one should require, that on the submanifold Q a two-form $\varpi(g_1, g_2)$ exists satisfying the relation

$$d\varpi(g_1, g_2) = \omega^{WZ}(g_1)|_Q - \omega^{WZ}(g_2)|_Q \,. \tag{1062}$$

To write the action of the WZW model with defect one should introduce an auxiliary disc D satisfying the conditions:

$$\partial B_1 = \Sigma_1 + D \;\text{ and }\; \partial B_2 = \Sigma_2 + \bar{D} \,, \tag{1063}$$

and continue the fields g_1 and g_2 on this disc always holding the condition (1061). After this preparations the topological part of the action takes the form :

$$S^{\text{top}-\text{def}} = \frac{k}{4\pi}\int_{B_1} \omega^{WZ}(g_1) + \frac{k}{4\pi}\int_{B_2} \omega^{WZ}(g_2) - \frac{k}{4\pi}\int_{D} \varpi(g_1, g_2) \,. \tag{1064}$$

Equation (1062) guarantees that (1064) is well defined.

The full action is

$$S = S^{\text{kin}} + S^{\text{top}-\text{def}} \tag{1065}$$

where

$$S^{\text{kin}-\text{def}}(g_1, g_2) = \frac{k}{4\pi}\int_{\Sigma_1} L^{\text{kin}}(g_1) d^2 z + \frac{k}{4\pi}\int_{\Sigma_2} L^{\text{kin}}(g_2) d^2 z \tag{1066}$$

Denote by C_μ a conjugacy class in group G:

$$C_\mu = \{h f_\mu h^{-1} = h e^{2i\pi\mu/k} h^{-1}, \quad h \in G\} \,, \tag{1067}$$

where $\mu \equiv \boldsymbol{\mu} \cdot \boldsymbol{H}$ is a highest weight representation integrable at level k, taking value in the Cartan subalgebra of the G Lie algebra.

Let us consider as the bibrane Q the submanifold:

$$(g_1, g_2) = (C_\mu p, p) \tag{1068}$$

or alternatively

$$g_1 g_2^{-1} = C_\mu \tag{1069}$$

We can easily check that the equation (1062) is satisfied with

$$\varpi(C,p) = \omega^{(f)}(h) - \mathrm{Tr}(C_\mu^{-1} dC_\mu dpp^{-1}) \tag{1070}$$

It is straightforward to prove that

$$\mathrm{Tr}(g_1^{-1}\delta g_1(g_1^{-1}dg_1)^2) - \mathrm{Tr}(g_2^{-1}\delta g_2(g_2^{-1}dg_2)^2) - \delta\varpi = dB_\mu\,. \tag{1071}$$

where

$$B_\mu = A_\mu(h) - \mathrm{Tr}(\delta pp^{-1}C^{-1}dC) + \mathrm{Tr}(C^{-1}\delta C dpp^{-1}) \tag{1072}$$

Recalling that the first two terms come from the equation

$$\delta\omega^{WZ} = d[\mathrm{Tr}(g^{-1}\delta g(g^{-1}dg)^2)]\,, \tag{1073}$$

we see that the existence of the one-form B satisfying (1071) is a consequence of the equation (1062).

The defect equation of motion is

$$\mathrm{Tr}\Big[\delta g_1 g_1^{-1}(\partial_z g_1 g_1^{-1} - \partial_{\bar z} g_1 g_1^{-1})\Big]d\tau - \mathrm{Tr}\Big[\delta g_2 g_2^{-1}(\partial_z g_2 g_2^{-1} - \partial_{\bar z} g_2 g_2^{-1})\Big]d\tau + B_\mu = 0 \tag{1074}$$

After some calculation one can show that (1074) implies:

$$J_1 = J_2 \qquad \text{and} \qquad \bar J_1 = \bar J_2 \tag{1075}$$

Overlap of the Defect operator with the coordinate wave function

To calculate the overlap with the boundary state, we will need the knowledge of S-matrix of $SU(2)$ at level k,

$$S_{aj} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{(2a+1)(2j+1)\pi}{k+2}\right)\,. \tag{1076}$$

In the large-k limit the ratio of S-matrix elements appearing in the boundary state simplifies to

$$\frac{S_{aj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)}\sin[(2j+1)\hat\psi]\,, \tag{1077}$$

where, to shorten the notation, we have introduced $\hat{\psi} = \frac{(2a+1)\pi}{k+2}$. Using these results, the overlap between the boundary state and the localised bulk state becomes

$$\langle\vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \sum_{j,m,n} \frac{(k+2)}{\pi} \sin[(2j+1)\hat{\psi}]\mathcal{D}^j_{nm}(g_1(\vec{\theta}_1))\mathcal{D}^j_{mn}(g_2^{-1}(\vec{\theta}_2))\,. \qquad (1078)$$

To simplify this expression we need the identity

$$\sum_m \mathcal{D}^j_{nm}(g_1(\vec{\theta}_1))\mathcal{D}^j_{mn'}(g_2^{-1}(\vec{\theta}_2)) = \mathcal{D}^j_{nn'}(g_1(\vec{\theta}_1)g_2^{-1}(\vec{\theta}_2))\,, \qquad (1079)$$

which follows from the fact that the matrices $\mathcal{D}^j_{nm}$ form a representation of the group. Finally, one needs the property of the Wigner D-functions that $\sum_n \mathcal{D}^j_{nn}(g) = \frac{\sin(2j+1)\psi}{\sin\psi}$, where ψ is the angle of the standard metric (756) and defined by the relation $\mathrm{Tr}g = 2\cos\psi$ (or in our case $\mathrm{Tr}(g_1 g_2^{-1}) = 2\cos\psi$). The overlap (1078) becomes

$$\langle\vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \frac{k+2}{\pi\sin\psi}\sum_j \sin[(2j+1)\hat{\psi}]\sin[(2j+1)\psi] \qquad (1080)$$

and from the completeness of $\sin(n\psi)$ on the interval $[0,\pi]$ one concludes

$$\langle\vec{\theta}_1|X_a|\vec{\theta}_2\rangle \sim \frac{k+2}{4\sin\psi}\delta(\psi-\hat{\psi})\,. \qquad (1081)$$

We see that the brane wave function is localised on $\psi =$ const. bulk states.